

LOCALLY HOMOGENEOUS PP-WAVES

WOLFGANG GLOBKE AND THOMAS LEISTNER

ABSTRACT. We show that every n -dimensional locally homogeneous pp-wave is a plane wave, provided it is indecomposable and its curvature operator, when acting on 2-forms, has rank greater than one. As a consequence we obtain that indecomposable, Ricci-flat locally homogeneous pp-waves are plane waves. This generalises a classical result by Jordan, Ehlers and Kundt in dimension 4. Several examples show that our assumptions on indecomposability and the rank of the curvature are essential.

1. BACKGROUND AND MAIN RESULTS

A semi-Riemannian manifold (\mathcal{M}, g) is *homogeneous* if it admits a transitive action by a group of isometries. This means that for each pair of points p and q in \mathcal{M} there is an isometry of (\mathcal{M}, g) that maps p to q . In the spirit of Felix Klein's *Erlanger Programm* to characterise geometries by their symmetry group, homogeneous manifolds are fundamental building blocks in geometry. Homogeneity is strongly tied to the geometry and the curvature of a manifold. For example, homogeneous Riemannian manifolds are geodesically complete, and, as an example for the link to curvature, we recall the celebrated result that any Ricci-flat homogeneous Riemannian manifold is flat [3]. A weaker version of homogeneity which still guarantees that the manifold looks the same everywhere is local homogeneity: a semi-Riemannian manifold is *locally homogeneous* if for each pair of points p and q in \mathcal{M} there is an isometry defined on a neighbourhood of p that maps p to q .

Here we will study local homogeneity for a certain class of Lorentzian manifolds, the so-called *pp-waves* and the *plane waves*. Locally, an $(n+2)$ -dimensional *pp-wave* admits coordinates $(x^-, x^1, \dots, x^n, x^+)$ such that

$$(1.1) \quad g := 2dx^+(dx^- + Hdx^+) + \delta_{ij}dx^i dx^j,$$

where $H = H(x^1, \dots, x^n, x^+)$ is a function not depending on x^- . For a *plane wave*, this function is required to be quadratic in the x^i 's with x^+ -dependent coefficients. In general, they are not homogeneous, but they admit a parallel null (i.e. non-zero and light-like) vector field. An invariant definition of pp-waves and plane waves is given as follows: A Lorentzian manifold (\mathcal{M}, g) is a *pp-wave* if it admits a parallel null vector field $V \in \Gamma(T\mathcal{M})$, i.e., $V \neq 0$, $g(V, V) = 0$ and $\nabla V = 0$, and if its curvature endomorphism $R : \Lambda^2 T\mathcal{M} \rightarrow \Lambda^2 T\mathcal{M}$ is non-zero and satisfies

$$(1.2) \quad R|_{V^\perp \wedge V^\perp} = 0,$$

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where $V^\perp := \{X \in T\mathcal{M} \mid g(X, V) = 0\}$. A *plane wave* is a pp-wave with the additional condition

$$(1.3) \quad \nabla_U R = 0 \quad \text{for all } U \in V^\perp.$$

Four-dimensional pp-waves were discovered in a mathematical context by Brinkmann [7] as one class of Einstein spaces that can be mapped conformally onto each other. In physics, plane waves and pp-waves appeared in general relativity [11], where they continue to play an important role (see for example [6, 17] for more references) as metrics for which the Einstein equations become linear and, when they solve these equations, describe the propagation of gravitational waves with flat surfaces as wave fronts. Later Penrose discovered that when “zooming in on null geodesics” every space-times has a plane wave as limit [22]. More recently, the conditions under which the homogeneity of a Lorentzian manifold is inherited by its Penrose limit were studied extensively by Figueroa-O’Farrill, Meessen and Philip [15, 23]. Moreover, having linear Einstein equations and a large number of parallel spinor fields, higher-dimensional plane waves and pp-waves recently appeared as supergravity backgrounds, e.g. in [16], and there is now a vast amount of literature on them. For more recent results on homogeneity see the work by Figueroa-O’Farrill et al. in [14, 12, 13].

A systematic study of 4-dimensional pp-waves was undertaken by Jordan, Ehlers and Kundt in [17] (see the English republication [18] and also [10], where the name *pp-wave* for *plane fronted with parallel rays* was introduced). Among other aspects, in [17] the isometries of 4-dimensional, gravitational (i.e. Ricci-flat) pp-waves are considered and the Killing equation is solved completely. As a consequence, the possible dimensions of the space of Killing vector fields are given and in each case the form of the metric is determined explicitly. This rather satisfying result allows [17] to conclude:

- (A) If a 4-dimensional Ricci-flat pp-wave (\mathcal{M}^4, g) is locally V^\perp -homogeneous, then it is a plane wave. In particular, if (\mathcal{M}^4, g) is Ricci-flat and locally homogeneous, then it is a plane wave.

Here, *local V^\perp -homogeneity* is a generalisation of local homogeneity taking into account the parallel null vector field V that exists on a pp-wave: the distribution V^\perp is parallel as well and defines a foliation of \mathcal{M} into totally geodesic leaves of codimension 1. Then we say that (\mathcal{M}, g) is *locally V^\perp -homogeneous* if for all pairs $p, q \in \mathcal{M}$ that are *in the same leaf* of V^\perp , there is a neighbourhood \mathcal{U} of p in \mathcal{M} and an isometry ϕ between (\mathcal{U}, g) and $(\phi(\mathcal{U}), g)$ that maps p to q .

Note that proving (A) amounts to showing that local homogeneity (in V^\perp -directions) forces all third derivatives $\partial_i \partial_j \partial_k H$, for $i, j, k = 1, \dots, n$ to vanish. This is a much harder problem in dimensions higher than 4. The methods used in [17] in order to solve the Killing equation are restricted to dimension 4 and also use that the function H is harmonic, as a consequence of Ricci-flatness.

Statement (A) is no longer true without the assumption of Ricci-flatness: Sippel and Goenner in [24] solved the Killing equation for a 4-dimensional pp-wave (\mathcal{M}^4, g) without assuming $\text{Ric} = 0$ and gave an example of a homogeneous pp-wave that is not a plane wave (see our Example 4.5). However, it turns out that the metric in this example decomposes into a product of a 3-dimensional pp-wave and \mathbb{R} . Note that in [17] such a decomposition was implicitly excluded by the Ricci-flatness: if a 4-dimensional

Ricci-flat manifolds splits as a Riemannian product, then it is flat. Hence, the results in [24, Table II, p. 1234] establish the following result:

- (B) If a 4-dimensional indecomposable pp-wave (\mathcal{M}^4, g) is locally V^\perp -homogeneous, then it is a plane wave. In particular, if (\mathcal{M}^4, g) is indecomposable and locally homogeneous, then it is a plane wave.

Here, when saying that the manifold is *indecomposable*, we mean that the holonomy algebra acts indecomposably. Therefore, when looking for a generalisation of (A) or (B) to arbitrary dimensions the notion of indecomposability is relevant. We say that a semi-Riemannian manifold (\mathcal{M}, g) is *strongly indecomposable* if (\mathcal{M}, g) does not split as a local semi-Riemannian product anywhere, i.e, there is no point in \mathcal{M} that has a neighbourhood on which g is a product metric. Clearly, by the local version of the de Rham-Wu splitting theorem, the holonomy algebra of a strongly indecomposable manifold acts indecomposably (i.e. without non-degenerate invariant subspace), but the converse in general is not true. In addition to strong indecomposability we will need another condition on the curvature tensor R of a pp-wave. From the very definition of a pp-wave it follows that the rank of R when acting on 2-forms does not exceed $\dim(\mathcal{M}) - 2$. When proving a generalisation of statement (B), we have to assume that generically the rank of R is larger than 1:

Theorem 1. *Let (\mathcal{M}, g) be a pp-wave of arbitrary dimension with parallel null vector field V . Assume that (\mathcal{M}, g) is strongly indecomposable and in addition that almost everywhere the rank of its curvature endomorphism acting on $\Lambda^2 T\mathcal{M}$ is larger than one. Then (\mathcal{M}, g) is a plane wave if it is locally V^\perp -homogeneous.*

Here by “almost everywhere” we mean that there is no open set on which the rank of the curvature endomorphism is ≤ 1 . Note that the assumption on the rank of the curvature prevents us from applying Theorem 1 to 3-dimensional pp-waves, for which the rank of R cannot be bigger than 1. Indeed, in Example 4.3 we exhibit a 3-dimensional, locally homogeneous pp-wave that is *not* a plane wave, which shows that the assumption on the rank is crucial. However, since Ricci-flat pp-waves always satisfy this assumption (see Lemma 3.3), we obtain a generalisation of statement (A) to arbitrary dimensions:

Corollary 1. *A strongly indecomposable, Ricci-flat and locally V^\perp -homogeneous pp-wave is a plane wave.*

In locally homogeneous manifolds all points have isometric neighbourhoods. Hence, a locally homogeneous manifold is strongly indecomposable whenever it is indecomposable, and the rank of the curvature endomorphism is constant. This yields

Corollary 2. *An indecomposable, locally homogeneous pp-wave is a plane wave if, at one point, the rank of the curvature endomorphism is greater than one.*

Corollary 3. *Indecomposable, Ricci-flat and locally homogeneous pp-waves are plane waves.*

Corollary 3 is an instance of the phenomenon that Ricci-flat pp-waves with some additional geometric conditions have to be plane waves. Another instance of this phenomenon is given in [21], where it is shown that compact Ricci-flat pp-waves are plane waves.

When proving Theorem 1 we use the following property which is implied by local V^\perp -homogeneity (see Lemma 2.3): for every $p \in \mathcal{M}$ there are Killing vector fields, defined on a neighbourhood of p , which, when evaluated at p , span $V^\perp|_p$. However, these Killing vector fields might not be local sections of V^\perp . If they are, we can drop the assumption on the rank of the curvature and obtain

Theorem 2. *Let (\mathcal{M}, g) be a strongly indecomposable pp-wave in which each point admits a neighbourhood \mathcal{U} with local Killing vector fields that span $V^\perp|_{\mathcal{U}}$. Then (\mathcal{M}, g) is a plane wave.*

This is a version of a result for *commuting* Killing vector fields tangent to V^\perp :

Theorem 3. *Let (\mathcal{M}, g) be a semi-Riemannian manifold of dimension m and assume that there are commuting Killing vector fields that span a null distribution (i.e., a distribution on which g degenerates) of rank $m - 1$. Then (\mathcal{M}, g) admits a parallel null vector field V and its curvature satisfies*

$$R(X, Y)Z = 0 \quad \text{and} \quad \nabla_X R = 0,$$

for all $X, Y, Z \in V^\perp$. In particular, if (\mathcal{M}, g) is Lorentzian, then it is a plane wave.

Jordan, Ehlers and Kundt [17, Theorem 4.5.2] proved Theorem 3 for 4-dimensional Lorentzian manifolds, but their proof works in any dimension and signature (see our Section 3). In contrast, our proofs of Theorems 1 and 2 use completely different methods than those in [17]. In fact, our proof of Theorem 1 does not require a full solution of the Killing equation (which we derive in Section 4) but a detailed analysis of its consequences (in Section 5). Moreover, we use algebraic results such as the classification of subalgebras of the Lie algebra of similarity transformations $\mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ of \mathbb{R}^n that act indecomposably on $\mathbb{R}^{1, n+1}$ via $\mathfrak{sim}(n) \subset \mathfrak{so}(1, n+1)$. This classification is due to Bérard-Bergery and Ikemakhen [4], and plays an important role in the classification of indecomposable Lorentzian holonomy algebras in [20].

As we have pointed out above, Example 4.3 shows that, at least in dimension 3, the condition on the rank of the curvature cannot be dropped. However, obvious generalisations of Example 4.3 lead either to *non-homogeneous* pp-waves (as in [10], see our Example 4.4) or to *decomposable* homogeneous pp-waves (as in [24], our Example 4.5). Hence, with statement (B) in mind, we are tempted to conjecture (see also Section 4.2 for more details):

Conjecture. *Any indecomposable locally homogeneous pp-wave of dimension larger than 3 is a plane wave.*

In relation to this we should point out that the rank assumption is independent from the assumption of strong indecomposability: in Example 3.2 we present a 4-dimensional, strongly indecomposable plane wave metric whose curvature has rank 1.

Locally homogeneous plane waves turn out to be reductive (see Section 4.3.3) and have been classified by Blau and O’Loughlin [5] (see our Section 4.3.2). As a consequence, with the exception of the curvature rank one case, our reduction to the plane waves yields a classification of indecomposable locally homogeneous pp-waves. The curvature rank one case remains open for further study. Also we believe that our methods employed in Section 5 are useful in a wider context and will give a better understanding of the more general class of indecomposable locally homogeneous Lorentzian manifolds.

The paper is structured as follows: In Sections 2 we review facts about locally homogeneous spaces. In Section 3 we present some facts about pp-waves and plane waves including a fundamental coordinate description. In Section 4 we derive the Killing equation for pp-waves in these coordinates (Theorem 4.1) and, moreover, use this to obtain some useful facts, including the reductivity of homogeneous plane waves. We also review the classification of homogeneous plane waves in [5] and give a couple of examples that illustrate important features. Finally, in Section 5 we will use the obtained results to prove Theorems 1 and 2. The appendix contains a proof of the coordinate description that turns out to be fundamental for our approach.

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2. KILLING VECTOR FIELDS AND LOCALLY HOMOGENEOUS SPACES

Let (\mathcal{M}, g) be a semi-Riemannian manifold with Levi-Civita connection ∇ . A *Killing vector field* $K \in \Gamma(T\mathcal{M})$ is a vector field whose flow ϕ_t consists of local isometries of g , i.e. $\phi_t : (\mathcal{U}, g) \rightarrow (\phi_t(\mathcal{U}), g)$ is an isometry, where \mathcal{U} is a neighbourhood of p on which ϕ_t is defined. If K is complete, then all ϕ_t 's are global isometries.

Clearly, K is a Killing vector field if and only if the $(2, 0)$ -tensor $g(\nabla K, \cdot)$ is skew-symmetric, i.e.

$$(2.1) \quad g(\nabla_X K, Y) + g(X, \nabla_Y K) = 0 \quad \text{for all } X, Y \in T\mathcal{M}.$$

Let us denote the real vector space of Killing vector fields of (\mathcal{M}, g) by \mathfrak{k} . The Lie bracket of vector fields equips \mathfrak{k} with a Lie algebra structure.

In order to derive the integrability conditions for the Killing equation (2.1), we recall the classical approach by Kostant [19]. Let us denote by $\mathfrak{so}(T\mathcal{M}, g) := \{\phi \in \text{End}(T\mathcal{M}) \mid g(\phi(X), Y) + g(\phi(Y), X) = 0\}$ the bundle of skew-symmetric endomorphisms. For a Killing vector field K , we define the section $\phi^K := \nabla K$ of $\mathfrak{so}(T\mathcal{M}, g)$. A straightforward computation shows that the Killing equation (2.1) implies that

$$\nabla_X \phi^K = -R(K, X),$$

where R denotes the curvature tensor of g defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$. Hence, we define the vector bundle

$$\mathcal{K} := T\mathcal{M} \oplus \mathfrak{so}(T\mathcal{M}, g) \longrightarrow \mathcal{M}$$

and furnish it with the covariant derivative

$$\nabla_X^{\mathcal{K}} \begin{pmatrix} K \\ \phi \end{pmatrix} := \begin{pmatrix} \nabla_X K - \phi(X) \\ \nabla_X \phi + R(K, X) \end{pmatrix}.$$

We get the vector space isomorphism

$$\mathfrak{k} \simeq \{\text{parallel sections of } (\mathcal{K}, \nabla^{\mathcal{K}})\},$$

which shows that $\dim(\mathfrak{k}) \leq \text{rk}(\mathcal{K}) = \frac{1}{2}m(m+1)$, where $m = \dim(\mathcal{M})$. It also shows that a Killing vector field K is uniquely determined by the values $K|_p \in T_p\mathcal{M}$ and

$\nabla K|_p \in \mathfrak{so}(T_p\mathcal{M}, g_p)$ at a point $p \in M$ and thus yields an injection of \mathfrak{k} into the Lie algebra of semi-Euclidean motions,

$$(2.2) \quad \begin{aligned} \kappa : \mathfrak{k} &\hookrightarrow \mathfrak{so}(r, s) \ltimes \mathbb{R}^{r, s} \\ K &\mapsto - \left((\varepsilon_i \varepsilon_j g_p(\nabla_{\mathbf{e}_i} K, \mathbf{e}_j))_{i, j=1}^m, (\varepsilon_k g_p(K_p, \mathbf{e}_k))_{k=1}^m \right), \end{aligned}$$

where (r, s) is the signature of g , $p \in \mathcal{M}$ and \mathbf{e}_i an orthonormal basis of $T_p\mathcal{M}$, i.e., $g(\mathbf{e}_i, \mathbf{e}_j) = \varepsilon_i \delta_{ij}$. Note the minus in front of the image. It ensures that for the flat metric on $\mathbb{R}^{r, s}$ this map is a Lie algebra isomorphism (instead of an *anti*-isomorphism) between the Killing vector fields and the group of motions. In general, this map is *not* a Lie algebra homomorphism. For example, the Killing vector fields of the m -sphere are isomorphic to $\mathfrak{so}(m+1)$ rather than $\mathfrak{so}(m) \ltimes \mathbb{R}^n$. In fact, a lengthy but straightforward computation reveals

$$\nabla[K, \hat{K}] = [\nabla K, \nabla \hat{K}] - R(K, \hat{K}),$$

where the right-hand side bracket is the commutator of linear maps, which yields

$$(2.3) \quad \kappa([K, \hat{K}]) - [\kappa(K), \kappa(\hat{K})]_{\mathfrak{so}(t, s) \ltimes \mathbb{R}^{t, s}} = - \left(\varepsilon_i \varepsilon_j R_p(K_p, \hat{K}_p, \mathbf{e}_i, \mathbf{e}_j), 0 \right).$$

Returning to the integrability condition for the Killing equation, we compute the curvature R^K of ∇^K and we get

$$R^K(X, Y) \begin{pmatrix} K \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ (\nabla_K R)(X, Y) - (\phi \cdot R)(X, Y) \end{pmatrix},$$

where $\phi \cdot R$ denotes the canonical action of an endomorphism on $(3, 1)$ -tensors. Hence the existence of a parallel section (K, ϕ) of \mathcal{K} gives the integrability condition

$$(2.4) \quad \nabla_K R = \phi \cdot R,$$

where $\phi = \nabla K$ and R is the curvature of g , i.e., we have for $X, Y, Z \in T\mathcal{M}$ that

$$(\nabla_K R)(X, Y)Z = \phi(R(X, Y)Z) - R(\phi(X), Y)Z - R(X, \phi(Y))Z - R(X, Y)\phi(Z).$$

Now, assume that (\mathcal{M}, g) enjoys the existence of a parallel vector field V . We define two vector spaces

$$\begin{aligned} \mathfrak{k}(V) &:= \{K \in \mathfrak{k} \mid g(K, V) = 0\}, \\ \mathfrak{k}'(V) &:= \{K \in \mathfrak{k} \mid \nabla_V K = 0\} \end{aligned}$$

and observe

Lemma 2.1. *If V is a parallel vector field, then we have the following inclusion of subalgebras*

$$\mathfrak{k}(V) \subset \mathfrak{k}'(V) \subset \mathfrak{k}.$$

Proof. First we check the inclusion $\mathfrak{k}(V) \subset \mathfrak{k}'(V)$. Indeed, for a Killing vector field $K \in \mathfrak{k}$, the derivative of the function $g(V, K)$ satisfies

$$(2.5) \quad X(g(K, V)) = g(\nabla_X K, V) = -g(\nabla_V K, X).$$

First, this implies that if $K \in \mathfrak{k}(V)$ then we also have $\nabla_V K = 0$, i.e., $K \in \mathfrak{k}'(V)$.

Next we note that both $\mathfrak{k}(V)$ and $\mathfrak{k}'(V)$ are subalgebras: Clearly, if V is parallel, V^\perp is involutive and hence $\mathfrak{k}(V)$ is closed under the bracket. Moreover, for $K, \hat{K} \in \mathfrak{k}'(V)$ we have that

$$\nabla_V[K, \hat{K}] = \nabla_V \nabla_K \hat{K} - \nabla_V \nabla_{\hat{K}} K = \nabla_{[K, V]} \hat{K} - \nabla_{[\hat{K}, V]} K = 0,$$

since $V \lrcorner R = 0$ and $[K, V] = -\nabla_V K = 0$. Hence, also $\mathfrak{k}'(V)$ is a subalgebra. \square

Lemma 2.2. *A Killing field K satisfies $K \in \mathfrak{k}(V)$ if and only if at some, and hence any, point $p \in \mathcal{M}$ we have*

$$(2.6) \quad g(K, V)|_p = 0, \quad g(\nabla_X K, V)|_p = 0 \quad \text{for all } X \in T_p \mathcal{M}.$$

Proof. First note that (2.5) implies that any $K \in \mathfrak{k}(V)$ satisfies $g(\nabla_X K, V) \equiv 0$.

Conversely, assume $g(K, V)|_p = 0$ and $g(\nabla_X K, V)|_p = 0$ for all $X \in T_p \mathcal{M}$ at $p \in \mathcal{M}$. Let γ be a geodesic emanating from p with $\dot{\gamma}(0) = X$. Then by (2.5) we have

$$\begin{aligned} \frac{d^2}{dt^2}(g(K, V)|_{\gamma(t)}) &= -g(\nabla_{\dot{\gamma}(t)} \nabla_V K, \dot{\gamma}(t)) \\ &= -g(\nabla_V \nabla_{\dot{\gamma}(t)} K, \dot{\gamma}(t)) - g(\nabla_{[\dot{\gamma}(t), V]} K, \dot{\gamma}(t)) \\ &= g(\nabla_{\dot{\gamma}(t)} K, \nabla_V \dot{\gamma}(t)) + g(\nabla_{\dot{\gamma}(t)} K, [\dot{\gamma}(t), V]) \\ &= 0, \end{aligned}$$

i.e., $g(K, V)|_{\gamma(t)}$ is linear in t . Hence it is determined by its value and its derivative at p which we both have assumed to be zero, forcing $g(K, V)|_{\gamma(t)} \equiv 0$. This shows that $g(K, V)$ is zero on a normal neighbourhood and thus zero everywhere. \square

This lemma implies the following: Let $v \in \mathbb{R}^{r,s}$ be the image of V in $\mathbb{R}^{r,s}$ under κ , i.e., $\kappa(V) = (0, v)$, and let $\mathfrak{stab}(v)$ be its stabiliser in $\mathfrak{so}(r, s)$. Then

$$\kappa : \mathfrak{k}(V) \hookrightarrow \mathfrak{stab}(v) \ltimes \mathbb{R}^{r,s}.$$

We will work with a different vector space of Killing vector fields, namely with

$$\mathfrak{k}_p(V) := \{K \in \mathfrak{k} \mid g(K, V)|_p = 0\},$$

for a fixed point $p \in \mathcal{M}$. In general, this is not a Lie algebra. However, we will see that for pp-waves it is a Lie algebra, a fact which turns out to be very useful.

Now we consider locally homogeneous and locally V^\perp -homogeneous manifolds as defined in Section 1. Both can be described in terms of Killing vectors.

(\mathcal{M}, g) is locally homogeneous if and only if for each point there exists Killing vector fields spanning $T_p \mathcal{M}$ when evaluated at p , i.e., for each point, the evaluation map combined with the projection on $\mathbb{R}^{r,s}$

$$\kappa : \mathfrak{k} \rightarrow \mathfrak{so}(r, s) \ltimes \mathbb{R}^{r,s} \rightarrow \mathbb{R}^{r,s}$$

is surjective. Moreover, (\mathcal{M}, g) is homogeneous (the isometry group acts transitively on \mathcal{M}) if and only if this holds for *complete* Killing vector fields.

Analogously, we have

Lemma 2.3. *Let V be a parallel vector field on (\mathcal{M}, g) . If (\mathcal{M}, g) is locally V^\perp -homogeneous, then for each $p \in \mathcal{M}$ there exist local Killing vector fields on a neighbourhood \mathcal{U} of p that span $V^\perp|_p$ when evaluated at p .*

Proof. Let $p \in \mathcal{M}$ and $X \in V^\perp|_p \subset T_p\mathcal{M}$. Let \mathcal{N}_p be a leaf of V^\perp through p and $\xi : (-\varepsilon, \varepsilon) \rightarrow \mathcal{N}_p$ a curve such that $\dot{\xi}(0) = X$. Then, by assumption, there is a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ in the Lie pseudo-group G of local isometries around p such that $\gamma_t(p) = \xi(t)$. Let $Y \in \mathfrak{g} = T_{\text{Id}}G$ be the tangent vector of γ at $t = 0$, i.e., $\dot{\gamma}(0) = Y$. Now let $\phi_t^Y := \exp(tY)$ be the one-parameter pseudo-group that is defined by Y . This allows us to define a vector field on a neighbourhood \mathcal{U} of p by

$$K(q) := \frac{d}{dt}(\phi_t^Y(q))|_{t=0},$$

with $q \in \mathcal{U}$. Since the flow of K is given by isometries, it is a Killing vector field. But we also have that $K(p) = X$ because

$$K(p) = \frac{d}{dt}(\exp(tY)(p))|_{t=0} = d\Psi|_{\text{Id}} \circ d\exp|_0(Y) = d\Psi|_{\text{Id}}(Y) = d\Psi|_{\text{Id}}(\dot{\gamma}(0)) = \dot{\xi}(0) = X,$$

where $\Psi : G \rightarrow \mathcal{M}$ is defined by $\Psi(g) = g(p)$ and we use that $d\exp|_0 = \text{Id}_{\mathfrak{g}}$. \square

In general, these Killing vectors do not have to be tangent to V^\perp everywhere.

Finally, note that a locally homogeneous manifold is strictly indecomposable (as defined in Section 1) whenever it is indecomposable (i.e., the holonomy algebra acts indecomposably, that is without non-degenerate invariant subspace): if a locally homogeneous manifold is a local product somewhere, it is a local product everywhere and hence the holonomy algebra has a non-degenerate invariant subspace.

Unfortunately this does not hold in the case of local V^\perp -homogeneity for a parallel null vector field V . This can be easily seen for pp-waves as in (1.1) on \mathbb{R}^{n+2} : here the leaves of V^\perp are given as $x^+ = c$ constant. If $H(x^1, \dots, x^n, x^+) \equiv 0$ for $x^+ \in (a, b)$ but $\det(\partial_i \partial_j (H)|_{(x^1, \dots, x^n, x^+)}) \neq 0$ for some other x^+ , then the holonomy algebra acts indecomposably. However, near a point with $x^+ \in (a, b)$ the metric is flat.

3. PP-WAVES AND PLANE WAVES

Here we recall some basic properties of pp-waves and plane waves as defined in Section 1. First note that the defining equation (1.2) is equivalent to

$$(3.1) \quad R(U, W) = 0 \quad \text{for all } U, W \in V^\perp,$$

or to

$$(3.2) \quad R(X, Y)U \in \mathbb{R}V \quad \text{for all } U \in V^\perp \text{ and } X, Y \in T\mathcal{M}.$$

A general pp-wave has an Abelian holonomy algebra contained in \mathbb{R}^n , where \mathbb{R}^n is an Abelian ideal in the stabiliser $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ in $\mathfrak{so}(1, n+1)$ of a null vector. The holonomy algebra is indecomposable if and only if it is equal to \mathbb{R}^n . A pp-wave has the following coordinate description:

Lemma 3.1. *Let (\mathcal{M}, g) be a pp-wave and let $p \in \mathcal{M}$. Then there are local coordinates $\varphi = (x^-, \mathbf{x} = (x^1, \dots, x^n), x^+)$ on a neighbourhood \mathcal{U} of p and a function $H \in C^\infty(\varphi(\mathcal{U}))$ such that $H = H(x^+, \mathbf{x})$ not depending on x^- such that,*

$$(3.3) \quad g = 2dx^+(dx^- + (H \circ \varphi)dx^+) + \delta_{ij}dx^i dx^j,$$

where δ_{ij} is the Kronecker symbol and where we use the summation convention. In these coordinates the parallel null vector field is given by $V|_{\mathcal{U}} = \partial_- := \frac{\partial}{\partial x^-}$. These coordinates are usually called Brinkmann coordinates after [7].

Moreover, these coordinates can be chosen such that $\varphi(p) = 0$ and

$$(3.4) \quad H(x^+, \mathbf{0}) \equiv 0, \quad \frac{\partial H}{\partial x^i}(x^+, \mathbf{0}) \equiv 0,$$

for all x^+ from an interval around zero. We call these coordinates normal Brinkmann coordinates centred at p .

Since the existence of coordinates as in (3.3) is well known, we only have to prove normality, i.e., the property (3.4). For sake of completeness we give a full proof of Lemma 3.1 but defer it to the appendix.

In Brinkmann coordinates the non-vanishing components of ∇ are

$$(3.5) \quad \begin{aligned} \nabla \partial_i &= \partial_i(H) dx^+ \otimes \partial_- \\ \nabla \partial_+ &= dH \otimes \partial_- - dx^+ \otimes \text{grad}(H), \end{aligned}$$

where $\text{grad}(H) = \delta^{ij} \partial_i(H) \partial_j$ denotes the gradient of H with respect to the flat metric $\delta_{ij} dx^i dx^j$ on \mathbb{R}^n . This property justifies the term “normal” in Lemma 3.1: the covariant derivatives vanish at $\mathbf{x} = \mathbf{0}$. The covariant derivatives of the corresponding 1-forms $dx^i = g(\partial_i, \cdot)$, $dx^+ = g(\partial_+, \cdot)$ and $dx^- = g(\partial_+ - 2H\partial_-, \cdot)$ are

$$\begin{aligned} \nabla dx^+ &= 0 \\ \nabla dx^i &= \partial_i H dx^+ \otimes dx^+ \\ \nabla dx^- &= -2dH dx^+. \end{aligned}$$

For a pp-wave the parallel null vector field V defines a parallel null distribution V^\perp of rank $n+1$ for which the connection induced by the Levi-Civita connection on the leaves of V^\perp is flat. In Brinkmann coordinates, each leaf is defined by $x^+ = c$ constant and parametrised by the coordinates x^-, x^1, \dots, x^n , and the formulae (3.5) show the flatness of the induced connection. Moreover, equations (3.5) imply that all the curvature components vanish apart from

$$(3.6) \quad R(\partial_i, \partial_+, \partial_j, \partial_+) = \partial_i \partial_j H,$$

and the components that are determined by this term via the symmetries of R . That is, we have

$$R = 4\partial_i \partial_j H (dx^i \wedge dx^+) (dx^i \wedge dx^+),$$

in which we use Einstein’s summation convention, and $\varphi \wedge \psi = \frac{1}{2}(\varphi \otimes \psi - \psi \otimes \varphi)$ and $\varphi \psi = \frac{1}{2}(\varphi \otimes \psi + \psi \otimes \varphi)$ are the alternating and the symmetric product of two tensors. Hence, the Ricci tensor of a pp-waves is given by

$$\text{Ric} = -\Delta H (dx^+),$$

where $\Delta = \sum_{i=1}^n \partial_i^2$ is the flat Laplacian. Moreover, the covariant derivative of R is

$$\nabla R = 4dH_{ij} (dx^i \wedge dx^+) (dx^i \wedge dx^+),$$

including the differentials of the functions $H_{ij} := \partial_i \partial_j H$. This shows that for a pp-wave to be a plane wave it requires $\partial_i \partial_j \partial_k H = 0$. Therefore, for a plane wave the function H is a quadratic polynomial in the x^i ’s, i.e., in normal Brinkmann coordinates we have

$$2H(x^+, \mathbf{x}) = \mathbf{x}^\top S(x^+) \mathbf{x}$$

where \mathbf{x} denotes the column vector (x^1, \dots, x^n) and $S(x^+)$ is a symmetric $n \times n$ -matrix depending on x^+ . Plane waves satisfy the vacuum Einstein equations, i.e., are Ricci-flat if and only if $S(x^+)$ is traceless for all x^+ .

A subclass of plane waves are the solvable Lorentzian symmetric spaces, called *Cahen-Wallach spaces* after [8]. As symmetric spaces, they satisfy $\nabla R = 0$ which forces the matrix S to be constant. If S has no trace, Cahen-Wallach spaces provide remarkable examples of Ricci-flat, non flat symmetric spaces, contrasting the Riemannian situation where Ricci-flat symmetric spaces are flat.

The relation (3.6) on a coordinate neighbourhood shows that the rank of R as an endomorphism of $\Lambda^2 T\mathcal{M}$ is equal to n if and only if $\det(\text{Hess}(H)) \neq 0$. Indeed, the rank is smaller than n if and only if there is a vector $X = \xi^i \partial_i \in V^\perp$ such that $R(X \wedge \partial_+) = 0$ which is equivalent to $R(X, \partial_+, \partial_j, \partial_+) = 0$ for all j , i.e., $\xi^i \partial_i \partial_j H = 0$. The curvature of a pp-wave and its derivatives are mapped into its holonomy algebra at p as follows, where we work with normal Brinkmann coordinates centred at p :

$$(\nabla_{X_1} \dots \nabla_{X_k} R)(\partial_i, \partial_+) \mapsto \begin{pmatrix} 0 & (X_1(\dots (X_k(\partial_i(\partial_j H \dots))|_0)_{j=1}^n) & 0 \\ 0 & 0 & \vdots \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that if there is *one point* where the Hessian of H has determinant not zero, then the pp-wave is indecomposable. However, the following example shows that the converse not true, i.e., there are indecomposable pp-waves, for which the rank of the curvature endomorphism is smaller than n *on an open set*.

Example 3.2. We give an example of a strongly indecomposable 4-dimensional plane wave whose curvature has a kernel everywhere, and which therefore has everywhere rank 1. Given two functions a_1 and a_2 on \mathbb{R} with $a_1^2 + a_2^2 \neq 0$ we consider the matrix

$$S = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{pmatrix}.$$

which has constant rank one. Then S defines a plane wave metric

$$g = 2dx^+(dx^- + \mathbf{x}^\top S(x^+) \mathbf{x} dx^+) + d\mathbf{x}^2.$$

Its curvature tensor is given by the matrix S and hence has everywhere rank 1. However the derivative of the curvature is given by the matrix

$$(\nabla_{\partial_+} R)(\partial_+, \partial_i, \partial_+, \partial_j) = \dot{a}_i a_j + a_i \dot{a}_j$$

which has determinant

$$\det(\dot{S}) = 4a_1 a_2 \dot{a}_1 \dot{a}_2 - (\dot{a}_1 a_2 + a_1 \dot{a}_2)^2 = -(\dot{a}_1 a_2 - a_1 \dot{a}_2)^2,$$

which in general is not zero. Therefore, as the first derivative of the curvature has no kernel, the holonomy of g is equal to \mathbb{R}^2 and hence g is strongly indecomposable.

We can even choose the matrix S in a way that the resulting plane wave is homogeneous. Indeed, if we set

$$S_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then

$$\exp(x^+ F) = \begin{pmatrix} \cos(x^+) & -\sin(x^+) \\ \sin(x^+) & \cos(x^+) \end{pmatrix}$$

and

$$S(x^+) = \exp(-x^+ F) S_- \exp(x^+ F) = \begin{pmatrix} \cos(x^+)^2 & -\cos(x^+) \sin(x^+) \\ -\cos(x^+) \sin(x^+) & \sin(x^+)^2 \end{pmatrix},$$

has constant rank one. According to Blau and O'Loughlin [5] (see also Section 4.3.2), S defines a homogeneous plane wave metric and

$$\det(\dot{S}) = -6 \cos(x^+)^2 \sin(x^+)^2 - \cos(x^+)^4 - \sin(x^+)^4 \neq 0$$

shows that it is indecomposable.

In order to deduce Corollaries 1 and 3 from Theorem 1 we observe

Lemma 3.3. *If a pp-wave is Ricci-flat and its curvature endomorphism has rank 1, then it is flat.*

Proof. Assume that the curvature endomorphism has rank 1 at a point p . This implies that there is an orthonormal basis V_p, E_1, \dots, E_n of $V^\perp|_p$ such that $R(E_+, E_i, E_+, E_j) = 0$ unless $i = j = 1$, where E_+ is transversal to $V^\perp|_p$. But then

$$0 = \text{Ric}(E_+, E_+) = \sum_{i=1}^n R(E_+, E_i, E_+, E_i) = R(E_+, E_1, E_+, E_1),$$

so g is flat. □

Regarding indecomposability, in what follows the following observation will be useful:

Lemma 3.4. *On a pp-wave (\mathcal{M}, g) , let \mathcal{U} be simply connected patch of Brinkmann coordinates and let $L = a^i \partial_i$ be a non-zero vector field on \mathcal{M} with constant coefficients a^i such that $R(X, Y)L = 0$ for all $X, Y \in T\mathcal{U}$. Then the holonomy of (\mathcal{U}, g) is properly contained in \mathbb{R}^n , i.e., it does not act indecomposably. Moreover, g is locally a product metric.*

Proof. Since $L = a^i \partial_i$ has constant coefficients and no ∂_+ -component, it is easy to see that its parallel transport along a curve γ is given as $P_\gamma(L|_{\gamma(0)}) = \lambda \partial_- + L|_{\gamma(1)}$ for some $\lambda \in \mathbb{R}$ depending on the curve. Since L as well as ∂_- are annihilated by the curvature tensor, we get that

$$R(X, Y) \circ P_\gamma(L) = R(X, Y)(\lambda \partial_- + L) = 0.$$

Using the Ambrose-Singer holonomy theorem, this shows that not only the null vector ∂_- but also the space-like vector L is invariant under the holonomy algebra of $(\mathcal{U}, g|_{\mathcal{U}})$, which, as a consequence is reduced from \mathbb{R}^n to a decomposable subalgebra. The reminder of the statement follows from the local version of the de Rham-Wu decomposition theorem. □

We conclude this section with a proof Theorem 3. It generalises the proof in [17] but avoids the use of coordinates.

Proof of Theorem 3. Let (\mathcal{M}, g) be a semi-Riemannian manifold of dimension $n+2$ and K_-, K_1, \dots, K_n commuting Killing vector fields such that K_- is null and the K_i are orthogonal to K_- . We will show that this implies that $V := K_-$ is parallel and that $R(X, Y)Z = 0$ and $\nabla_X R = 0$ whenever $X, Y, Z \in V^\perp$. First note that we have

$$\begin{aligned} g(\nabla_{K_i} K_j, K_k) &= -g(\nabla_{K_k} K_j, K_i) = -g(\nabla_{K_j} K_k, K_i) = g(\nabla_{K_i} K_k, K_j) \\ &= g(\nabla_{K_k} K_i, K_j) = -g(\nabla_{K_j} K_i, K_k) = -g(\nabla_{K_i} K_j, K_k), \end{aligned}$$

and hence

$$(3.7) \quad g(\nabla_{K_i} K_j, K_k) = 0$$

for $i, j, k = 0, \dots, n$. Set $g_{ij} := g(K_i, K_j)$. Clearly, $g_{0i} = 0$ but the Koszul formula also gives

$$(3.8) \quad dg_{jk}(K_i) = g(\nabla_{K_i} K_j, K_k) + g(\nabla_{K_k} K_i, K_j) = 0.$$

Now we show that $V = K_-$ is parallel. To this end fix a null vector field Z such that $g(V, Z) = 1$ and $g(Z, K_i) = 0$ for $i = 1, \dots, n$. Clearly we have $g(\nabla_Z K_i, Z) = 0$, and the Koszul formula also gives us that

$$\begin{aligned} 0 &= g(\nabla_Z K_i, K_j) + g(\nabla_{K_j} K_i, Z) \\ &= \frac{1}{2} (Z(g_{ij}) - Z(g_{ij}) + g([Z, K_i], K_j) + g([K_j, Z], K_i) + g([Z, K_i], K_j) + g([Z, K_j], K_i)) \\ &= g([Z, K_i], K_j). \end{aligned}$$

This implies that

$$g(\nabla_Z K_i, K_j) = -g(\nabla_{K_j} K_i, Z) = \frac{1}{2} Z(g_{ij}),$$

and in particular that $\nabla_Z V = 0$ and $\nabla_{K_i} V = 0$, i.e., that $V = K_-$ is parallel. Moreover, we obtain that

$$\nabla_{K_i} K_j = -\frac{1}{2} Z(g_{ij}) V.$$

This implies that

$$\begin{aligned} (3.9) \quad 2R(K_i, K_j)K_k &= (K_j(Z(g_{ik})) - K_i(Z(g_{jk}))) V \\ &= ([K_j, Z](g_{ik}) - [K_i, Z](g_{jk})) V \\ &= 0, \end{aligned}$$

because of (3.8) and since the equation $0 = g([Z, K_i], V)$ from above shows that $[Z, K_i]$ has no Z -component. Hence, we have shown that $R(K_i, K_j)K_k = 0$, i.e., that g is a pp-wave in the case when g is Lorentzian. In order to show that $\nabla_X R = 0$ for all $X \in V^\perp$ we use the integrability condition (2.4). Denote by $\phi_i := \nabla K_i$. Obviously $\phi_- = 0$ and $\phi_i(K_j) = -\frac{1}{2} Z(g_{ij}) V$ and $\phi_i(Z) \in \text{span}(K_i)_{i=0}^n$. This and (3.9) together with the integrability condition (2.4) gives us

$$\nabla_{K_i} R = \phi_i \cdot R = 0,$$

and hence the statement of Theorem 3. \square

4. THE KILLING EQUATION FOR PP-WAVES

4.1. The Killing equation in normal Brinkmann coordinates. Here we derive the Killing equation in Brinkmann coordinates and then specialise this to normal Brinkmann coordinates found in Lemma 3.1. Mostly we follow [5] where the Killing equation for plane waves in Brinkmann coordinates is derived and solved. We fix Brinkmann coordinates $(x^-, \mathbf{x} = (x^1, \dots, x^n), x^+)$ and, using (3.5), compute the Lie derivative $\mathcal{L}_K g$ of the metric g in direction of a vector field

$$K := K^- \partial_- + K^i \partial_i + K^+ \partial_+,$$

as

$$\begin{aligned} \frac{1}{2} \mathcal{L}_K g &= \partial_- K^+ (dx^-)^2 + \delta_{ij} \partial_k K^i dx^k dx^j + \left(\dot{K}^- + K^i H_i + K^+ \dot{H} + 2H \dot{K}^+ \right) (dx^+)^2 \\ &\quad + (\delta_{ij} \partial_- K^j + \partial_i K^+) dx^- dx^i + \left(\partial_i K^- + \dot{K}^i + 2H \partial_i K^+ \right) dx^i dx^+ \\ &\quad + \left(\partial_- K^- + 2H \partial_- K^+ + \dot{K}^+ \right) dx^- dx^+, \end{aligned}$$

where we write $H_i := \partial_i H$, $H^i := \delta^{ij} H_j$, $\dot{H} := \partial_+ H$, and in general a dot for ∂_+ derivatives. Hence, K is a Killing vector field if and only if its components satisfy the following system

$$(4.1) \quad \partial_- K^+ = 0$$

$$(4.2) \quad \partial_i K^j + \partial_j K^i = 0$$

$$(4.3) \quad \dot{K}^- + K^i H_i + K^+ \dot{H} + 2H \dot{K}^+ = 0$$

$$(4.4) \quad \partial_- K^i + \partial_i K^+ = 0$$

$$(4.5) \quad \partial_i K^- + \dot{K}^i + 2H \partial_i K^+ = 0$$

$$(4.6) \quad \partial_- K^- + \dot{K}^+ = 0$$

These equations were derived in [5] and in the following we review some of the arguments given there. Because of (4.1), K^+ is independent of x^- . Hence, differentiating (4.4) and (4.6) with respect to x^- gives

$$\partial_-^2 K^i = \partial_-^2 K^- = 0,$$

showing that K^- and all K^i are linear in x^- , whereas differentiating (4.2) with respect to x^- and (4.4) with respect to x^j and symmetrising over i and j gives

$$0 = 2\partial_i \partial_j K^+$$

for all i, j showing that K^+ is linear in the x^i 's. Hence, there are functions $\alpha^+, \alpha_1, \dots, \alpha_n$ of x^+ only such that

$$K^+ = \alpha_i x^i + \alpha^+.$$

Then equation (4.6) becomes

$$0 = \partial_- K^- + \partial_+ K^+ = \partial_- K^- + \dot{\alpha}_i x^i + \dot{\alpha}^+,$$

and hence, there is a function $A^- := A^-(x^+, x^1, \dots, x^n)$ depending on (x^+, x^1, \dots, x^n) such that

$$K^- = -(\dot{\alpha}_i x^i + \dot{\alpha}^+) x^- + A^-.$$

Furthermore, equation (4.4) becomes

$$0 = \partial_- K^i + \alpha_i$$

yielding the existence of functions $A^i = A^i(x^+, x^1, \dots, x^n)$ depending on (x^+, x^1, \dots, x^n) such that

$$K^i = -\alpha_i x^- + A^i.$$

With this information at hand, we evaluate (4.5) and get

$$0 = -2\dot{\alpha}_i x^- + \partial_i A^- + \dot{A}^i + 2H\alpha_i.$$

Since A^- , A^i and H are independent of x^- this shows that the α_i 's are constant, i.e., $\alpha_i \equiv a_i \in \mathbb{R}$. Hence, K is a Killing vector field if and only if its components are given as

$$\begin{aligned} K^+ &= a_i x^i + \alpha^+ \\ K^- &= -\dot{\alpha}^+ x^- + A^- \\ K^i &= -a_i x^- + A^i \end{aligned}$$

for constants a_i , a function α^+ of x^+ and functions A^- and A^i of (x^1, \dots, x^n, x^+) subject to the equations

$$(4.7) \quad -(\ddot{\alpha}^+ + a_i H^i) x^- + \dot{A}^- + A^i H_i + (a_i x^i + \alpha^+) \dot{H} + 2H\dot{\alpha}^+ = 0$$

$$(4.8) \quad \partial_i A^j + \partial_j A^i = 0$$

$$(4.9) \quad \partial_i A^- + \dot{A}^i + 2H a_i = 0.$$

Differentiating (4.7) with respect to x^- and then with respect to x^i we obtain

$$(4.10) \quad a_i \partial_j \partial^i H = 0.$$

Recalling formula (3.6), this shows that the vector field $L = a^i \partial_i$ on \mathcal{M} , for $a^i := a_i$ constants, is annihilated by the curvature tensor R of g , i.e., $R(X, Y)L = 0$ for all $X, Y \in T\mathcal{M}$.

From now on we will assume that (\mathcal{M}, g) is strongly indecomposable, i.e., that the holonomy algebra of $(\mathcal{U}, g|_{\mathcal{U}})$ acts indecomposably. Under this assumption, Lemma 3.4 implies by (4.10) that all the constants a_i vanish,

$$a_i = 0.$$

Differentiating equation (4.7) with respect to x^- yields that $\alpha^+ = ax^+ + b$ is linear.

Now, with the a_i being zero, differentiating equation (4.8) with respect to x^+ and equation (4.9) with respect to x^j and symmetrising over i and j gives us

$$\partial_i \partial_i A^- = 0,$$

which shows that A^- is linear in the x^i 's. Plugging this back into (4.9), and differentiating with respect to x^j yields

$$0 = \partial_j \dot{A}^i,$$

which shows that A^i is of the form $A^i = \psi^i + F^i$, where the ψ^i are functions of x^+ only and F^i are functions of (x^1, \dots, x^n) . Consequently, there is a function φ of x^+ such that

$$A^-(x^+, \mathbf{x}) = -\mathbf{x}^\top \dot{\Psi} + \varphi.$$

where we write $\Psi = (\psi^1, \dots, \psi^n)$.

Finally, the functions F^i are subject to the Euclidean Killing equation

$$\partial_i F^j + \partial_j F^i = 0,$$

the solutions of which are given, up to constants, by a skew-symmetric matrix $f^i_j = -f^j_i$ such that $F^i = f^i_j x^j$. Plugging all this back into equation (4.3) we obtain that any Killing vector field K on an indecomposable pp-wave (\mathcal{M}, g) in Brinkmann coordinates is of the form

$$(4.11) \quad K(x^-, x^+, \mathbf{x}) = -\left(ax^- + \varphi(x^+) + \mathbf{x}^\top \dot{\Psi}(x^+)\right) \partial_- + (\Psi(x^+) + F\mathbf{x})^i \partial_i + (ax^+ + b) \partial_+,$$

where a , b and $F = (f^i_j) \in \mathfrak{so}(n)$ are constant, and φ and $\Psi = (\psi^1, \dots, \psi^n)$ are functions of x^+ satisfying the equation

$$(4.12) \quad -\ddot{\Psi}^\top \mathbf{x} - \dot{\varphi} + \text{grad}(H)^\top (\Psi + F\mathbf{x}) + (ax^+ + b)\dot{H} + 2aH = 0$$

Now, in normal Brinkmann coordinates, we can simplify equation (4.12):

Theorem 4.1. *Let (\mathcal{M}, g) be a strongly indecomposable pp-wave, $p \in \mathcal{M}$, and let $(\mathcal{U}, (x^+, x^-, \mathbf{x} = (x^1, \dots, x^n)))$ be normal Brinkmann coordinates centred at p with $2H := g(\partial_+, \partial_+)$. Then K is a Killing vector field if and only if*

$$(4.13) \quad K = (c - ax^- - \dot{\Psi}^\top \mathbf{x}) \partial_- + (\Psi + F\mathbf{x})^i \partial_i + (ax^+ + b) \partial_+,$$

where $a, b, c \in \mathbb{R}$, $F \in \mathfrak{so}(n)$ are constant and $\Psi \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ subject to the Killing equation

$$(4.14) \quad \ddot{\Psi}^\top \mathbf{x} - \text{grad}(H)^\top (\Psi + F\mathbf{x}) - (ax^+ + b)\dot{H} - 2aH = 0.$$

Moreover, for the commutator $\hat{K} = [K_1, K_2]$ of two Killing vector fields K_1, K_2 the parameters are

$$(4.15) \quad \begin{aligned} \hat{a} &= 0 \\ \hat{b} &= a_2 b_1 - a_1 b_2 \\ \hat{c} &= \dot{\Psi}_1^\top \Psi_2 - \Psi_1^\top \dot{\Psi}_2 - a_1 c_2 + a_2 c_1 \\ \hat{F} &= -[F_1, F_2] \\ \hat{\Psi} &= F_2 \cdot \Psi_1 - F_1 \cdot \Psi_2 + (a_1 x^+ + b_1) \dot{\Psi}_2 - (a_2 x^+ + b_2) \dot{\Psi}_1. \end{aligned}$$

Proof. Clearly, K in (4.13) is a Killing vector field as its components satisfy equation (4.12) with $\varphi(x^+) \equiv -c$.

On the other hand, we have seen that every Killing vector field in Brinkmann coordinates is of the form (4.11) with components satisfying equation (4.12). Choosing the Brinkmann coordinates to be normal at p , equation (4.12) when taken along $\mathbf{x} = \mathbf{0}$ becomes $\dot{\varphi} \equiv 0$, which we solve by $\varphi(x^+) \equiv -c$.

Finally, it is a matter of checking that the induced Lie bracket is of the form (4.15). Note that, as required, the term $\dot{\Psi}_1^\top \Psi_2 - \Psi_1^\top \dot{\Psi}_2$ is constant as a consequence of both Ψ_1 and Ψ_2 being solutions of equation (4.19). \square

Let us make a few observations. The fact that c does not appear in (4.14) is due to ∂_- , as a parallel vector field, is a Killing vector field. Moreover, the parameters a, b, c, F, Ψ uniquely determine the Killing vector field K , which is determined by the values of its covariant derivative at the point p . For the covariant derivatives of K we compute

$$(4.16) \quad \begin{aligned} \nabla_{\partial_-} K &= -a\partial_- \\ \nabla_{\partial_i} K &= -\left(\dot{\psi}^i - (ax^+ + b)\partial_i H\right)\partial_- + f_i^k \partial_k \\ \nabla_{E_+} K &= \left(\dot{\psi}^i - (ax^+ + b)\partial_i H\right)\partial_i + aE_+, \end{aligned}$$

where $E_+ = \partial_+ - H\partial_-$ and we have to use the Killing equation (4.12) to obtain the last derivative. Hence, at zero, the Killing vector in (4.11) and its covariant derivative is given by

$$(4.17) \quad \begin{aligned} K|_0 &= c\partial_- + \psi^i(0)\partial_i + b\partial_+ \\ \nabla_{\partial_-} K|_0 &= -a\partial_- \\ \nabla_{\partial_i} K|_0 &= -\dot{\psi}^i(0)\partial_- + f_i^k \partial_k \\ \nabla_{\partial_+} K|_0 &= \dot{\psi}^i(0)\partial_i + a\partial_+ \end{aligned}$$

Moreover, differentiating equation (4.14) yields

$$(4.18) \quad \ddot{\Psi} + F \operatorname{grad}(H) - \operatorname{Hess}(H)(\Psi + F\mathbf{x}) - (ax^+ + b) \operatorname{grad}(\dot{H}) - 2a \operatorname{grad}(H) = 0.$$

By the properties of the normal Brinkmann coordinates from Lemma 3.1, this becomes a second order linear ODE system for $\Psi = (\psi^1, \dots, \psi^n)$ when taken along $\mathbf{x} = \mathbf{0}$:

$$(4.19) \quad \ddot{\Psi}(t) - \operatorname{Hess}(H)(t, \mathbf{0})\Psi(t) = 0.$$

Fixing initial conditions $\Psi(0)$ and $\dot{\Psi}(0)$ gives a unique solution to this system. This illustrates how K is completely determined by the initial conditions.

In the remainder of the section we will consider some special cases, known results and examples.

4.2. Transversal Killing vector fields. We will see that a crucial issue of the Killing equation on pp-waves is the existence of Killing vector fields that are transversal to the parallel null distribution V^\perp of rank $n+1$.

First note that, if $\dot{H} = 0$, then there is always the transversal Killing vector field ∂_+ , but in general transversal Killing vector fields are much harder to find and the situation is much more involved. For example, for certain pp-waves there exist Killing vector fields with $b = 0$ but $a \neq 0$ being tangent to V^\perp *only* along the leaf $x^+ = 0$ but transversal elsewhere, i.e., pp-waves for which

$$\mathfrak{k}(V) := \{K \in \mathfrak{k} \mid g(K, V) = 0\}$$

and

$$\mathfrak{k}_p(V) := \{K \in \mathfrak{k} \mid g(K, V)|_p = 0\}$$

are different. Note that Theorem 4.1 and formulae (4.16) show that

$$\mathfrak{k}' := \{K \in \mathfrak{k} \mid \nabla_V K = 0\}$$

and its subalgebra $\mathfrak{k}(V)$ are actually ideals in the Lie algebra \mathfrak{k} of Killing vector fields. In fact we have that $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}'$. Killing vector fields that are transversal *at some point* project onto non-zero elements in the quotient Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$.

Corollary 4.2. *The Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$ is isomorphic to a subalgebra of $\mathfrak{aff}(1)$, the Lie algebra of affine transformations of \mathbb{R} . In particular, if $\mathfrak{k}/\mathfrak{k}(V)$ is 2-dimensional, then there are two Killing vector fields K and \hat{K} such that*

$$\begin{aligned} K &= x^+ \partial_+ \mod V^\perp \\ \hat{K} &= \partial_+ \mod V^\perp. \end{aligned}$$

Proof. The theorem shows that there is a Lie algebra homomorphism

$$\mathfrak{k} \ni (ax^+ + b)\partial_+ + K^i \partial_i + K^- \partial_- \mapsto (a, b) \in \mathfrak{aff}(1),$$

the kernel of which is $\mathfrak{k}(V)$. Hence $\mathfrak{k}/\mathfrak{k}(V)$ injects homomorphically into $\mathfrak{aff}(1)$. If $\mathfrak{k}/\mathfrak{k}(V)$ is 2-dimensional, we can invert this map obtaining two Killing vector fields of the required form. \square

Example 4.3. Here we will give an example of a 3-dimensional pp-wave for which the Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$ is indeed 2-dimensional, and more importantly, which is *locally homogeneous but not a plane wave*, showing that the assumption on the curvature in Theorem 1 is essential. Consider the pp-wave (\mathcal{M}, g) where $\mathcal{M} = \mathbb{R}^3$ and

$$g = 2dx^+(dx^- + e^{2ax}dx^+) + dx^2,$$

where $a \in \mathbb{R} \setminus \{0\}$ is a constant and (x^+, x^-, x) are the standard coordinates in \mathbb{R}^3 . In particular, the function $H(x^+, x)$ is

$$H(x) = e^{2ax}.$$

Since $\frac{\partial H}{\partial x^+} = 0$, the Killing equation (4.14) takes the form

$$\ddot{\psi}x - 2ae^{2ax}\psi - 2ae^{2ax} = 0.$$

Solving this equation, we find that, in addition to $V = \partial_-$ and ∂_+ , there is another Killing vector field of g , namely

$$K = ax^+ \partial_+ - ax^- \partial_- - \partial_x.$$

Hence, \mathfrak{k} is 3-dimensional. Since $g(K, V) = ax^+$, we have $\mathfrak{k}(V) = \mathbb{R} \cdot \partial_-$ and thus $\dim(\mathfrak{k}/\mathfrak{k}(V)) = 2$. Moreover, the Killing vector fields span the tangent space $T_p \mathcal{M}$ at any point $p \in \mathcal{M}$, so (\mathcal{M}, g) is a locally homogeneous pp-wave. However, (\mathcal{M}, g) is strongly indecomposable since

$$R(\partial_x, \partial_+) = \begin{pmatrix} 0 & 2ae^{2ax} & 0 \\ 0 & 0 & -2ae^{2ax} \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

for any $x \in \mathbb{R}$, and (\mathcal{M}, g) clearly is *not a plane wave* since

$$(\nabla_{\partial_x} R)(\partial_x, \partial_+) = \begin{pmatrix} 0 & 4a^2 e^{2ax} & 0 \\ 0 & 0 & -4a^2 e^{2ax} \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Example 4.4 (Ehlers & Kundt). Similar examples with $\dim(\mathfrak{k}/\mathfrak{k}(V)) = 2$ but in dimension 4 are given by Ehlers and Kundt in [10, Table 2-5.1] as a correction to [17]. For one class of examples H is given as the real part of the complex function

$$e^{2az}, \quad \text{with } a > 0,$$

of $z = x^1 + ix^2$. Then ∂_- and ∂_+ and

$$-a(x^-\partial_- + x^+\partial_+) - \partial_1$$

span the Killing vector fields. For the other class, H is given as the real part of

$$e^{2ia \ln(z)}, \quad \text{with } a \neq 0.$$

Here, the Killing vector fields are spanned by ∂_- and ∂_+ and

$$-a(x^-\partial_- + x^+\partial_+) + x^1\partial_2 - x^2\partial_1.$$

Note that with $\dim(\mathfrak{k}) = 3$ and $\dim(\mathfrak{k}_p(V)) = 2$ both metrics are neither homogeneous nor V^\perp -homogeneous.

Example 4.5 (Sippel & Goenner). Another example of this type with $\dim(\mathfrak{k}/\mathfrak{k}(V)) = 2$ in dimension 4 was given by Sippel and Goenner in [24, Table II, no. 9]. These examples are pp-wave metrics on \mathbb{R}^4 which are locally homogeneous but not plane waves. However, they turn out to be *decomposable*. The pp-wave metric is defined by

$$H(x^1, x^2) := c e^{a_1 x^1 - a_2 x^2},$$

with c, a_1, a_2 constants with $a_1^2 + a_2^2 \neq 0$. The Killing vector fields are given by ∂_- , ∂_+ and

$$\begin{aligned} K &:= x^+(a_2\partial_1 + a_1\partial_2) + (a_2x^1 + a_1x^2)\partial_- \in \mathfrak{k}(\partial_-), \\ K_i &:= \partial_i + a_i(x^+\partial_+ - x^-\partial_-), \end{aligned}$$

for $i = 1, 2$, and span the tangent space. However a coordinate transformation

$$x = a_1x^1 - a_2x^2, \quad y = a_2x^1 + a_1x^2$$

reveals that this metric is decomposable.

For plane waves we can show

Proposition 4.6. *Let (\mathcal{M}, g) be a strongly indecomposable plane wave. Then*

$$(4.20) \quad \dim(\mathfrak{k}/\mathfrak{k}(V)) \leq 1.$$

Proof. Assume there are two linearly independent Killing vector fields that are not tangent to V^\perp . They are of the form

$$\begin{aligned} K &= x^+\partial_+ + (\psi + F\mathbf{x})^k\partial_k + K^-\partial_- \\ \hat{K} &= \partial_+ + (\hat{\psi} + \hat{F}\mathbf{x})^k\partial_k + \hat{K}^-\partial_-. \end{aligned}$$

Now, differentiating equation (4.18) again we obtain

$$(4.21) \quad (\psi + F\mathbf{x})^k\partial_k \text{Hess}(H) + [F, \text{Hess}(H)] + (ax^+ + b)\text{Hess}(\dot{H}) + 2a\text{Hess}(H) = 0.$$

For a plane wave in normal Brinkmann coordinates with $S = \text{Hess}(H)$ we have that $\partial_k S = 0$ and thus when taking equation (4.21) along $\mathbf{x} = \mathbf{0}$, we obtain for K and \hat{K} that

$$\begin{aligned} [F, S] - x^+\dot{S} - 2S &= 0 \\ [\hat{F}, S] - \dot{S} &= 0. \end{aligned}$$

This implies that

$$[F - x^+\hat{F}, S] - 2S = 0,$$

for all x^+ . Since the map $S \mapsto [F - x^+ \hat{F}, S]$ when acting on symmetric matrices is skew-symmetric with respect to the trace form, which, on the other hand, is positive definite on symmetric matrices, we obtain that $S \equiv 0$, which is a contradiction. \square

A fundamental question is whether, in dimensions greater than 3, (V^\perp) homogeneity and inecomposability forces $\mathfrak{k}/\mathfrak{k}(V)$ to have dimension 1. Because of the additional term $\partial_k \text{Hess}(H)$, we are not able to prove (4.20) for arbitrary (V^\perp) homogeneous pp-waves, but we conjecture that it is true:

Conjecture 4.7. For an indecomposable locally homogeneous pp-wave of dimension greater than 3, the Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$ is at 1-dimensional.

Our proof of Theorem 1 in Section 5 will show that if this conjecture is true, then in dimensions greater than 3 we can drop the assumption on the curvature in Corollary 2 (see Remark 5.8).

4.3. Plane waves. In this section we will recall some facts about plane waves, for which the Killing equation is completely solved in [5].

4.3.1. Plane waves and the Heisenberg algebra. For a plane wave defined by a matrix $S(x^+)$ the Lie algebra $\mathfrak{k}(V)$ always contains the Heisenberg algebra $\mathfrak{he}(n)$. Indeed, for a plane wave we have

$$H = \frac{1}{2} \mathbf{x}^\top S(x^+) \mathbf{x}$$

for a symmetric x^+ -dependent matrix S , and hence

$$\text{grad}(H) = S\mathbf{x}, \quad \text{Hess}(H) = S.$$

For such H , multiplying the differentiated equation (4.18) by \mathbf{x} implies the Killing equation (4.14), which therefore becomes equivalent to (4.18). On the other hand, when setting $F = 0$ and $a = b = 0$, equation (4.18) is equivalent to the linear ODE system (4.19) which, for a plane wave, becomes

$$(4.22) \quad \ddot{\Psi} - S\Psi = 0.$$

Hence, we have Killing vector fields

$$(4.23) \quad \begin{aligned} L_i &:= \phi_i^k \partial_k - \mathbf{x}^\top \dot{\Phi}_i \cdot \partial_- \\ K_i &:= \psi_i^k \partial_k - \mathbf{x}^\top \dot{\Psi}_i \cdot \partial_-, \end{aligned}$$

where $\Phi_i = (\phi_i^k)_{k=1,\dots,n}$ and $\Psi_i = (\psi_i^k)_{k=1,\dots,n}$ are solutions to the linear ODE system (4.22) with initial conditions

$$\begin{aligned} \Phi_i(0) &= \mathbf{0}, & \dot{\Phi}_i(0) &= \mathbf{e}_i \\ \Psi_i(0) &= \mathbf{e}_i, & \dot{\Psi}_i(0) &= \mathbf{0}, \end{aligned}$$

which span $\mathfrak{he}(n)$. Clearly, ∂_- commutes with the K_i 's and L_j 's and we have

$$(4.24) \quad [L_i, K_j] = (\Phi_i^\top \dot{\Psi}_j - \Psi_j^\top \dot{\Phi}_i) \partial_- = -\delta_{ij} \partial_-$$

because the term $\Phi_i^\top \dot{\Psi}_j - \Psi_j^\top \dot{\Phi}_i$ is constant as a consequence of equation (4.19).

Clearly, for plane waves, there are *commuting* Killing vector fields $X_1, \dots, X_n, \partial_-$ spanning the null distribution V^\perp . Theorem 3 shows that this can *only* happen for plane waves.

4.3.2. *Homogeneous plane waves.* For plane waves, the Killing equation (4.18) becomes the following matrix ODE:

$$(4.25) \quad [S(x^+), F] + (ax^+ + b)\dot{S}(x^+) + 2aS(x^+) = 0.$$

In Section 4.3.1 we saw that \mathfrak{k} always contains a Heisenberg algebra. Now, for a plane wave to be locally homogeneous, we need an additional Killing vector field K transversal to $V^\perp|_p$. Hence, when working with normal Brinkmann coordinates centred at p , one has to find a solution of equation (4.25) with $b \neq 0$. This was done by Blau and O'Loughlin in [5]. Depending on a being zero or not, they found two families of homogeneous plane waves, where the metrics in both families are determined by the choice of a constant symmetric matrix S_- and a constant skew-symmetric matrix F .

In the first case, when $a = 0$ we can assume $b = 1$ and hence the Killing equation (4.25) just becomes

$$[S(x^+), F] + \dot{S}(x^+) = 0.$$

Clearly this is solved by

$$S(x^+) = e^{x^+F} S_- e^{-x^+F}$$

with a constant skew symmetric matrix F and a constant symmetric matrix S_- . Hence, the metrics in the first family are of the form

$$(4.26) \quad g = 2dx^+dx^- + (\mathbf{x}^\top e^{x^+F} S_- e^{-x^+F} \mathbf{x})(dx^+)^2 + d\mathbf{x}^2.$$

When defined on all of \mathbb{R}^{n+2} they are geodesically complete (see for example results by Candela, Flores and Sánchez [9, Prop. 3.5]).

In the second case we have $a \neq 0$ so that we can assume $a = 1$. Here the Killing equation (4.25) becomes an ODE with singularity at $x^+ = -b$,

$$(x^+ + b)\dot{S}(x^+) + [S(x^+), F] + 2S(x^+) = 0.$$

It has the solution

$$S(x^+) = \frac{1}{(x^+ + b)^2} (e^{\log(x^+ + b)F} S_- e^{\log(-(x^+ + b))F}),$$

again for constant (skew) symmetric matrices F and S_- . Hence, homogeneous plane wave metrics in the second family are of the form

$$(4.27) \quad g = 2dx^+dx^- + \frac{1}{(x^+ + b)^2} (\mathbf{x}^\top e^{\log(x^+ + b)F} S_- e^{\log(-(x^+ + b))F} \mathbf{x})(dx^+)^2 + d\mathbf{x}^2,$$

for constants F , S_- and b . They are only defined for $x^+ > -b$ and hence geodesically incomplete. Clearly, metrics for different b can be pulled back by a translation $x^+ \mapsto x^+ + b$ to the metric with $b = 0$ on $\{x^+ > 0\}$. Hence, metrics with different b are isometric to each other.

4.3.3. *Reductivity of homogeneous plane waves.* Here we will show that homogeneous plane waves are always *reductive*. This means that for some subalgebra \mathfrak{k}_0 of \mathfrak{k} generating a (locally) transitive group action, the stabiliser $\mathfrak{h} := \{K \in \mathfrak{k}_0 \mid K|_p = 0\}$ in \mathfrak{k}_0 of a point p has a vector space complement \mathfrak{m} in \mathfrak{k}_0 with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Proposition 4.8. *Homogeneous plane waves are reductively homogeneous.*

Proof. For a homogeneous plane wave, we take \mathfrak{k}_0 to be the subalgebra generated by the Killing fields

$$K_+, \partial_-, K_1, \dots, K_n, L_1, \dots, L_n,$$

where K_i, L_j are defined in (4.23) and $K_+ = -ax^-\partial_- + (F\mathbf{x})^i\partial_i + (ax^+ + b)\partial_+$ for a certain $F = (f_i^j) \in \mathfrak{so}(n)$ is transversal to V^\perp , which exists for homogeneous plane waves according to [5, (2.42)]. Working at p with normal Brinkmann coordinates centred at p , we see that \mathfrak{h} is spanned by the L_i 's defined in (4.23). Then the \mathfrak{h} -invariant complement \mathfrak{m} is spanned by ∂_-, K_+ and the n Killing vector fields

$$M_i := [K_+, L_i].$$

Note that this implies that

$$M_i|_p = b\phi_i^k(0)\partial_k|_p = b\partial_i|_p$$

Hence, since also $K_+|_p = b\partial_+|_p$, the vector space \mathfrak{m} defined in this way is indeed a complement to \mathfrak{h} . Moreover, since both M_i and L_j are tangent to V^\perp and without rotational component we obtain from (4.15) that

$$[L_j, M_i] = c\partial_-$$

for a constant c . Therefore we have $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ and the plane wave is reductive. \square

4.3.4. Cahen-Wallach spaces. For Cahen-Wallach spaces, the matrix $S(x^+)$ is constant and thus equation (4.25) can always be solved by setting $F = 0$ and $a = 0$ and thus yielding a Killing vector field transversal to V^\perp . For Cahen-Wallach spaces generically the algebra of Killing vector fields contains the oscillator algebra $\mathbb{R} \ltimes \mathfrak{he}(n)$ and hence has dimension at least $2n + 2$. The stabiliser algebra is equal to the holonomy algebra which is \mathbb{R}^n . A Cahen-Wallach space may have additional Killing vector fields in addition to $\mathbb{R} \ltimes \mathfrak{he}(n)$. In fact, the additional Killing vector fields are isomorphic to the centraliser in $\mathfrak{so}(n)$ of the constant matrix S defining the Cahen-Wallach space. Hence, it might have at most $\frac{1}{2}n(n-1)$ additional symmetries.

4.4. Dimension four. In [17] the Killing equation (4.14) for 4-dimensional pp-waves is explicitly solved under the assumption that (\mathcal{M}, g) is Ricci-flat, i.e., that H is harmonic, so that methods from complex analysis can be used. In particular, in [17, table on p. 79], the dimension of the space of Killing vector fields of a 4-dimensional, indecomposable, Ricci-flat pp-wave have been determined as $\dim(\mathfrak{k}) = 1, 2, 3, 5, 6$, and the metrics are explicitly given for each case. Moreover, in [24] the assumption of Ricci-flatness was dropped and new algebras of dimension 5, 6 and 7 appeared, almost reaching the upper bound of 8 we will deduce from Theorem 4.1 in Corollary 5.2. Further results about symmetries of 4-dimensional pp-waves were obtained in [1, 2].

5. PROOF OF THE MAIN RESULTS

In this section we will draw the conclusions from Theorem 4.1 that eventually will lead to a proof of Theorem 1. We assume that (\mathcal{M}, g) is an indecomposable pp-wave with parallel null vector field V . First we note:

Corollary 5.1. *Each Killing vector field satisfies $\nabla_V K \in \mathbb{R}V$.*

Proof. Let $p \in \mathcal{M}$ be an arbitrary point and chose normal Brinkmann coordinates centred at p (Lemma 3.1). In these coordinates a Killing vector field K is of the form (4.13) with its covariant derivative given in (4.16). Since $V = \partial_-$ on the coordinate patch, we get $\nabla_V K = a \cdot V$. \square

Now denote by \mathfrak{k} the Killing vector fields of (\mathcal{M}, g) . We describe the evaluation map κ at a point $p \in \mathcal{M}$ at which we choose a basis

$$(E_-, E_1, \dots, E_i, E_+)$$

of $T_p \mathcal{M}$ such that

$$g_p(E_-, E_+) = 1, \quad g_p(E_i, E_j) = \delta_{ij},$$

where $i, j = 1, \dots, n$, and all other $g_p(E_\alpha, E_\beta) = 0$ for $\alpha, \beta \in \{-, +, 1, \dots, n\}$. Moreover, in the proofs we will use normal Brinkmann coordinates centred at p and such that

$$(5.1) \quad E_- = \partial_-|_p, \quad E_i = \partial_i|_p, \quad E_+ = (\partial_+ - H\partial_-)|_p = \partial_+|_p.$$

In Theorem 4.1 we have seen that, for a Killing vector field K there are real numbers $a, b, c, X^i, Y^i, F = (f_i^j) \in \mathfrak{so}(n)$ such that

$$(5.2) \quad \begin{aligned} K|_p &= cE_- + X^i E_i + bE_+ \\ \nabla_{E_-} K|_p &= -aE_- \\ \nabla_{E_i} K|_p &= -Y_i E_- + f_i^k E_k \\ \nabla_{E_+} K|_p &= Y^i E_i + aE_+. \end{aligned}$$

Furthermore, we write $Y = (Y_i)$, $X^\top = (X_i)$ for the row vectors and $X = (X^i)$, $Y^\top = (Y^i)$ for the column vectors.

If we denote by $v \in \mathbb{R}^{1,n+1}$ the null vector that is the image of V under the evaluation map κ , i.e. $\kappa(V) = (0, v) \in \mathfrak{so}(1, n+1) \ltimes \mathbb{R}^{1,n+1}$, by Corollary 5.1 we have

$$\phi \in \mathfrak{stab}(\mathbb{R}v) \subset \mathfrak{so}(1, n+1)$$

for $\phi = \nabla K$. This stabiliser is equal to the Lie algebra of similarity transformations of \mathbb{R}^n ,

$$\begin{aligned} \mathfrak{stab}(\mathbb{R}v) &= \mathfrak{sim}(n) \\ &= (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n = \left\{ \left(\begin{pmatrix} a & u^\top & 0 \\ 0 & F & -u \\ 0 & 0 & -a \end{pmatrix} \mid \begin{array}{l} a \in \mathbb{R} \\ F \in \mathfrak{so}(n) \\ u \in \mathbb{R}^n \end{array} \right) \right\}, \end{aligned}$$

which is the minimal parabolic subalgebra in $\mathfrak{so}(1, n+1)$. Hence we obtain

Corollary 5.2. *The evaluation map κ in (2.2) is an injective vector space homomorphism*

$$(5.3) \quad \begin{aligned} \kappa : \mathfrak{k} &\hookrightarrow \mathfrak{sim}(n) \ltimes \mathbb{R}^{1,n+1} \\ K &\mapsto \left(\begin{pmatrix} a & Y & 0 \\ 0 & -F & -Y^\top \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} -c \\ -X \\ -b \end{pmatrix} \right) \end{aligned}$$

In particular,

$$1 \leq \dim(\mathfrak{k}) \leq (2n+3) + \frac{1}{2}n(n-1).$$

Unfortunately, the map in (5.3) is not a Lie algebra homomorphism. In fact, a direct computation using the bracket formula (4.15) confirms the observation (2.3) in the general setting and yields

$$(5.4) \quad \begin{aligned} [\kappa(K), \kappa(\hat{K})] - \kappa([K, \hat{K}]) &= R(K, \hat{K}, \partial_i, \partial_+)|_p \\ &= \left(\begin{pmatrix} 0 & (bS\hat{X} - \hat{b}SX)^\top & 0 \\ 0 & 0 & \hat{b}SX - bS\hat{X} \\ 0 & 0 & 0 \end{pmatrix}, 0 \right), \end{aligned}$$

where $S = \text{Hess}(H)|_p$. As a remedy, we consider the vector space

$$\mathfrak{k}_p(V) = \{K \in \mathfrak{k} \mid g(K, V)|_p = 0\}.$$

According to Theorem 4.1, when using coordinates of Lemma 3.1 around p elements in $\mathfrak{k}_p(V)$ are characterised by the condition $b = 0$. Hence, consulting formula (4.15) for the Lie bracket of two Killing vector fields, we make the following observation

Corollary 5.3. *$\mathfrak{k}_p(V)$ is a Lie subalgebra of \mathfrak{k} . Moreover, the evaluation map at p , when restricted to $\mathfrak{k}_p(V)$ is an injective Lie algebra homomorphism*

$$\begin{aligned} \kappa : \mathfrak{k}_p(V) &\hookrightarrow \mathfrak{co}(n) \ltimes \mathfrak{he}(n) \\ K &\mapsto \begin{pmatrix} a & -Y & c \\ 0 & F & X \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

where $\mathfrak{co}(n) := \mathbb{R} \oplus \mathfrak{so}(n)$ denotes the conformal Lie algebra and $\mathfrak{he}(n)$ the $(2n+1)$ -dimensional Heisenberg algebra.

Proof. That the evaluation map κ at p becomes a Lie algebra monomorphism follows from observation (2.3) and the defining property of pp-waves, which ensures that $R(K, \hat{K}, \cdot, \cdot)|_p = 0$ whenever $K, \hat{K} \in \mathfrak{k}_p(V)$. It can also be seen immediately from Theorem 4.1, $b = 0$ or from the observation (5.4). Moreover, if $b = 0$, the image of K_p lies in v^\perp , i.e., $\kappa(\mathfrak{k}_p(V)) \subset \mathfrak{sim}(n) \ltimes v^\perp$. Hence it remains to establish that

$$\begin{aligned} \mathfrak{sim}(n) \ltimes v^\perp &\simeq \mathfrak{co}(n) \ltimes \mathfrak{he}(n) \\ \left(\begin{pmatrix} a & Y^\top & 0 \\ 0 & F & -Y \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} c \\ X \\ 0 \end{pmatrix} \right) &\mapsto \begin{pmatrix} a & Y^\top & c \\ 0 & F & X \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

is indeed a Lie algebra isomorphism. But this is a straightforward computation. \square

Because of Lemma 2.1, for the subalgebra

$$\mathfrak{k}(V) = \{K \in \mathfrak{k} \mid g(K, V) = 0\}$$

of $\mathfrak{k}_p(V)$, whose elements are characterised by $a = b = 0$, we obtain

Corollary 5.4. *The evaluation map at p , when restricted to $\mathfrak{k}(V)$ is an injective Lie algebra homomorphism*

$$\begin{aligned} \kappa : \mathfrak{k}(V) &\hookrightarrow \mathfrak{so}(n) \ltimes \mathfrak{he}(n) \\ K &\longmapsto \begin{pmatrix} 0 & -Y & c \\ 0 & F & X \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Returning to the evaluation map, we note that the Lie algebra $\mathfrak{co}(n) \ltimes \mathfrak{he}(n)$ contains an Abelian ideal

$$\mathfrak{a} := \left\{ \begin{pmatrix} a & Y & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid Y \in \mathbb{R}^n, a \in \mathbb{R}, c \in \mathbb{R} \right\} \subset \mathfrak{co}(n) \ltimes \mathfrak{he}(n).$$

Therefore, the quotient $(\mathfrak{co}(n) \ltimes \mathfrak{he}(n))/\mathfrak{a}$ is a Lie algebra which turns out to be isomorphic to the Lie algebra of Euclidean motions $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ via

$$\begin{aligned} (\mathfrak{co}(n) \ltimes \mathfrak{he}(n))/\mathfrak{a} &\simeq \mathfrak{so}(n) \ltimes \mathbb{R}^n \\ \begin{pmatrix} a & Y^\top & c \\ 0 & F & X \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{a} &\mapsto \begin{pmatrix} F & X \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, we obtain

Corollary 5.5. *The evaluation map κ induces a Lie algebra homomorphism $\lambda : \mathfrak{k}_p(V) \rightarrow \mathfrak{so}(n) \ltimes \mathbb{R}^n$ given by*

$$\begin{aligned} \mathfrak{k}_p(V) &\xrightarrow{\lambda} \mathfrak{so}(n) \ltimes \mathbb{R}^n \\ K &\mapsto \begin{pmatrix} -F & -X \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Moreover, if $\mathfrak{k}_p(V)$ at p spans $V^\perp|_p$, then $\mathfrak{g} := \lambda(\mathfrak{k}_p(V)) \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$ is a subalgebra that acts indecomposably on $\mathbb{R}^{1,n+1}$ via

$$\begin{pmatrix} 0 & X^\top & 0 \\ 0 & -F & -X \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Since there are Killing vector fields that span $V^\perp|_p$, by the definition of λ for the projection $\text{pr}_{\mathbb{R}^n} : \mathfrak{so}(n) \ltimes \mathbb{R}^n \rightarrow \mathbb{R}^n$ onto the translations we have that

$$\text{pr}_{\mathbb{R}^n}(\lambda(\mathfrak{k}_p(V))) = \mathbb{R}^n.$$

This implies that $\mathfrak{g} = \lambda(\mathfrak{k}_p(V))$ acts indecomposably on $\mathbb{R}^{1,n+1}$. □

For the Killing vector fields $\mathfrak{k}(V)$ that are tangent to V^\perp we consider the ideal

$$\mathfrak{b} := \mathbb{R}^n \ltimes \mathbb{R} \subset \mathfrak{so}(n) \ltimes \mathfrak{he}(n),$$

for which we have $(\mathfrak{so}(n) \ltimes \mathfrak{he}(n))/\mathfrak{b} \simeq \mathfrak{so}(n) \ltimes \mathbb{R}^n$. In Corollary 5.4, this ideal corresponds to the elements

$$\begin{pmatrix} 0 & -Y & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In complete analogy to Corollary 5.5 we obtain from Corollary 5.4 the following

Corollary 5.6. *The evaluation map κ induces a Lie algebra homomorphism $\lambda : \mathfrak{k}(V) \rightarrow \mathfrak{so}(n) \ltimes \mathbb{R}^n$. Moreover, if $\mathfrak{k}(V)$ spans V^\perp , then $\mathfrak{h} := \lambda(\mathfrak{k}(V)) \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$ is a subalgebra that acts indecomposably on $\mathbb{R}^{1,n+1}$ as in Corollary 5.5.*

For the proof of Theorem 1 we will need a description of subalgebras of $\mathfrak{sim}(n)$ that act indecomposably on $\mathbb{R}^{1,n+1}$. Fortunately, there is such a classification due to Bérard-Bergery and Ikemakhen [4]:

Proposition 5.7. *Let $\mathfrak{g} \subset \mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ act indecomposably on $\mathbb{R}^{1,n+1}$. Then either \mathfrak{g} contains the translations \mathbb{R}^n , or \mathfrak{g} contains \mathbb{R}^q for $1 < q < n$, in which case there is a subalgebra $\mathfrak{h} \subset \mathfrak{so}(q)$ and a surjective linear map $\varphi : \mathfrak{h} \rightarrow \mathbb{R}^{n-q}$ such that \mathfrak{g} is of the form*

$$(5.5) \quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & X & \varphi(F) & 0 \\ 0 & F & 0 & -X \\ 0 & 0 & 0 & -\varphi(F) \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| F \in \mathfrak{h}, X \in \mathbb{R}^q \right\}.$$

The important property in this proposition is that the rotational part F of a transitively acting group of similarity transformations acts only on \mathbb{R}^p and annihilates the corresponding translational part $\varphi(F)$.

With this at hand we are ready to prove Theorem 1.

Proof of Theorem 1. By the defining property (1.3) of a plane wave, we have to show that at each point $p \in \mathcal{M}$ we have $\nabla_U R|_p = 0$ for all $U \in V^\perp|_p$. Working with a basis of the form (5.1), from the formulae for the curvature and the Levi-Civita connection of a pp-wave it follows that the only possibly non-vanishing terms of ∇R are $\nabla_{E_+} R(E_+, E_i, E_+, E_j)$ and

$$(5.6) \quad \begin{aligned} \nabla_k R_{ij} &:= \nabla_{E_k} R(E_+, E_i, E_+, E_j) \\ &= \nabla_{E_i} R(E_+, E_k, E_+, E_j) = \nabla_{E_j} R(E_+, E_k, E_+, E_i), \end{aligned}$$

for $i, j, k = 1, \dots, n$ and, because of the Bianchi identity, being symmetric in those. We will now use the integrability condition (2.4) to show that this term also vanishes. Because of our assumption that the curvature has rank greater than 1 almost everywhere, it suffices to work at a $p \in \mathcal{M}$ at which the rank of R is greater than 1. This just means that the rank of the matrix

$$R_{ij} := R(E_+, E_i, E_+, E_j)$$

is greater than 1.

Since there are Killing vector fields that span $V^\perp|_p$, we can apply Corollary 5.5 and Proposition 5.7 to $\mathfrak{g} = \lambda(\mathfrak{k}_p(V))$ giving two possible cases for \mathfrak{g} . In the first case, \mathfrak{g} contains the translations \mathbb{R}^n , i.e., there are Killing vector fields K_1, \dots, K_n with

$$\lambda(K_k) = \begin{pmatrix} 0 & \mathbf{e}_k^\top & 0 \\ 0 & 0 & -\mathbf{e}_k \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sim}(n).$$

By the definition of λ and recalling (5.2), this implies $K_k|_p = E_k$ and for the $\phi_k = \nabla K_k|_p$ that

$$(5.7) \quad \begin{aligned} \phi_k(E_j) &\in \mathbb{R}V_p, \quad \text{for } j = 1, \dots, n \\ \phi_k(E_+) &= a_k \partial_+ \mod V^\perp|_p, \end{aligned}$$

for $k = 1, \dots, n$. Without loss of generality we may assume that all but one a_i are equal to zero. Indeed, the linear map from $\text{span}(K_i)_{i=1}^n$ to \mathbb{R} defined by assigning a_i to each

K_i has a kernel of dimension at least $n - 1$. Hence, we can chose at least $n - 1$ linearly independent Killing vector fields in its kernel and possible one that is transversal to the kernel. The latter can be chosen in a way that, at p , it is orthogonal to the kernel, whereas the ones in the kernel can be chosen to be orthonormal to each other at p .

Hence, we can assume that $a_1 = \dots = a_{n-1} = 0$, and the integrability condition (2.4) becomes

$$(5.8) \quad \begin{aligned} \nabla_k R_{ij} &= R(\phi_k(E_+), E_i, E_+, E_j) + R(E_+, \phi_k(E_i), E_+, E_j) \\ &\quad + R(\phi_k(E_+), E_j, E_+, E_i) + R(E_+, \phi_k(E_j), E_+, E_i) \\ &= 2a_k R_{ij}, \end{aligned}$$

for $i, j, k = 1, \dots, n$. Therefore, we get

$$\nabla_k R_{ij} = 0,$$

for $k = 1, \dots, n - 1$ and $i, j = 1, \dots, n$, as well as

$$2a_n R_{ki} = \nabla_n R_{ki} = 0$$

for all $i = 1, \dots, n$ and $k = 1, \dots, n - 1$. Hence, if a_n was not zero, R_{nn} would be the only non-vanishing component of R_{ij} which contradicts the assumption that its rank is greater than one. Hence, also $a_n = 0$ and therefore $\nabla_k R_{ij} = 0$ for all i, j, k .

This gives us an idea how to proceed in the remaining case, in which \mathfrak{g} does not contain \mathbb{R}^n , but only an \mathbb{R}^N , for $1 < N < n$. Here, according to Proposition 5.7, \mathfrak{g} is of the form (5.5). In the following, we will use indices $A, B, C \dots \in \{1, \dots, N\}$ and $b, c, d, \dots \in \{N + 1, \dots, n\}$ and $i, j, k \in \{1, \dots, n\}$. For such \mathfrak{g} 's we have N Killing vector fields such that

$$\lambda(K_A) = \begin{pmatrix} 0 & \mathbf{e}_A^\top & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{e}_A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(n) \ltimes \mathbb{R}^n,$$

with

$$(5.9) \quad \begin{aligned} K_A|_p &= E_A \\ \phi_A(E_-) &= a_A \partial_- \\ \phi_A(E_i) &\in \mathbb{R}V_p, \quad \text{for } i = 1, \dots, n \\ \phi_A(E_+) &= a_A \partial_+ \mod V^\perp|_p, \end{aligned}$$

and $n - N$ Killing vector fields K_b , with

$$\lambda(K_b) = \begin{pmatrix} 0 & 0 & \mathbf{e}_b^\top & 0 \\ 0 & \overset{(b)}{F} & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{e}_b \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(n) \ltimes \mathbb{R}^n.$$

Note that by Proposition 5.7 all the $\overset{(b)}{F} \in \mathfrak{so}(N)$ are non-zero. By the definition of λ and looking at (5.2) this implies for $\phi_b = \nabla K_b|_p$ that

$$\begin{aligned}
 (5.10) \quad & K_b|_p = E_b \\
 & \phi_b(E_-) = -a_b \partial_- \\
 & \phi_b(E_A) = \overset{(b)}{f}_A^B \mod \mathbb{R}V_p \\
 & \phi_b(E_c) \in \mathbb{R}V_p \\
 & \phi_b(E_+) \in a_b \partial_+ \mod V^\perp|_p.
 \end{aligned}$$

As before, without loss of generality, we can assume that a_N and a_n are the only a_i 's that are possibly non-zero. Then we have

$$(5.11) \quad \nabla_A R_{ij} = 2a_A R_{ij}$$

$$(5.12) \quad \nabla_b R_{cd} = 2a_b R_{cd}$$

$$(5.13) \quad \nabla_b R_{cA} = 2a_b R_{cA} + \overset{(b)}{f}_A^B R_{cB}$$

$$(5.14) \quad \nabla_b R_{AB} = 2a_b R_{AB} + 2 \overset{(b)}{f}_{(A}^C R_{B)C}$$

With our assumption $a_1 = \dots a_{N-1} = a_{N+1} = \dots = a_{n-1} = 0$ equation (5.11) gives

$$(5.15) \quad \nabla_A R_{ij} = 0, \text{ for all } A \neq N$$

and thus

$$(5.16) \quad a_N R_{Aj} = 0, \text{ for all } A \neq N.$$

Similarly, equation (5.12) yields

$$(5.17) \quad \nabla_b R_{cd} = 0, \text{ for all } b \neq n$$

and thus

$$(5.18) \quad a_n R_{bc} = 0, \text{ for all } (b, c) \neq (n, n).$$

Furthermore, using the total symmetry of $\nabla_i R_{jk}$ we observe that equation (5.13) gives

$$(5.19) \quad 2a_A R_{bc} = 2a_b R_{cA} + \overset{(b)}{f}_A^B R_{cB}$$

and (5.14) yields

$$(5.20) \quad 2a_A R_{Bc} = 2a_c R_{AB} + \overset{(c)}{f}_{(A}^D R_{B)D}.$$

With all these relations, the total symmetry of $\nabla_i R_{jk}$ implies that the only possibly non-vanishing terms of $\nabla_i R_{jk}$ are

$$\begin{aligned}
 (5.21) \quad & \nabla_N R_{NN} = 2a_N R_{NN} \\
 & \nabla_n R_{nn} = 2a_n R_{nn} \\
 & \nabla_N R_{nN} = a_N R_{nN} = 2a_n R_{NN} + \overset{(n)}{f}_N^C R_{NC} \\
 & \nabla_n R_{nN} = a_n R_{nn} = 2a_n R_{nN} + \overset{(n)}{f}_N^B R_{nB}.
 \end{aligned}$$

Now we consider two cases: First assume that $a_N \neq 0$. In this case equation (5.16) implies that

$$(5.22) \quad R_{Aj} = 0 \quad \text{for all } A \neq N$$

Evaluating (5.19) for $A = N$ yields

$$(5.23) \quad 2a_N R_{bc} = 2a_b R_{cN} + \overset{(b)}{f}_N^B R_{cB} = 2a_b R_{cN} = 2a_c R_{bN}$$

since $\overset{(b)}{F}$ is skew and hence $\overset{(b)}{f}_N^N = 0$. Evaluating this for $b \neq n$ we get that

$$(5.24) \quad R_{bc} = 0, \quad \text{for all } (b, c) \neq (n, n).$$

Moreover, equation (5.20) for $A = B = N$ for $c \neq n$ gives

$$2a_N R_{Nc} = \overset{(c)}{f}_N^D R_{ND} = 0$$

again because of (5.22) and the skew-symmetry of $\overset{(c)}{F}$. So we get

$$(5.25) \quad R_{Nb} = 0 \quad \text{for } b \neq n.$$

Putting (5.22), (5.24) and (5.25) together we get that R_{NN} , R_{nn} and R_{Nn} are the only non vanishing components of R_{ij} . According to the last two equations of (5.21) they are related by

$$\begin{aligned} a_N R_{nN} &= a_n R_{NN} \\ a_N R_{nn} &= a_n R_{nN} \end{aligned}$$

This implies that $a_n \neq 0$ because otherwise R_{NN} would be the only non-vanishing component of R_{ij} which contradicts to the rank of R_{ij} being greater than one. But this implies

$$a_n a_N \det \begin{pmatrix} R_{NN} & R_{Nn} \\ R_{nN} & R_{nn} \end{pmatrix} = 0,$$

which finally leads a contradiction to the rank of R_{ij} being greater than one.

It remains to derive a contradiction in the case when $a_N = 0$. If also $a_n = 0$ we are done, so we assume $a_n \neq 0$. In this case (5.18) implies that

$$(5.26) \quad R_{bc} = 0, \quad \text{for all } (b, c) \neq (n, n).$$

Moreover (5.19) for $b = n$ implies that each $(R_{cB})_{B=1}^N$ is an eigenvector of $\overset{(n)}{F}$. Since $a_n \neq 0$ is real and $\overset{(n)}{F}$ skew, this implies that $R_{cB} = 0$ for all c and B .

Moreover equation (5.20) for $c = n$ becomes

$$-2a_n R_{AB} = \overset{(n)}{f}_A^D R_{BD} + \overset{(n)}{f}_B^D R_{AD} = \overset{(n)}{f}_A^D R_{BD} - \overset{(n)}{f}_D^B R_{AD}$$

which just means that the matrix (R_{AB}) is an eigenvector with eigenvalue $-2a_n$ for the adjoint action of $\overset{(n)}{F} \in \mathfrak{so}(n)$ on the symmetric matrices, i.e.,

$$(5.27) \quad -2a_n R = [\overset{(n)}{F}, R].$$

Since $\overset{(n)}{F}$, when acting on symmetric matrices via the commutator, is skew-symmetric with respect to the trace form, which, on the other hand, is positive definite on symmetric matrices, (5.27) implies $R_{AB} = 0$. Hence, again R_{nn} is the only non-vanishing component of R_{ij} which contradicts our assumption that the rank of the curvature endomorphism is larger than one. This concludes the proof of Theorem 1. \square

This proof and Corollary 5.4 immediately give us a **proof of Theorem 2** when taking into account that Killing vector fields from $\mathfrak{k}(V)$ have $a_i = 0$ for $i = 1, \dots, n$.

Remark 5.8. Note that our proof shows that for indecomposable homogeneous pp-waves with 1-dimensional Lie algebra $\mathfrak{k}/\mathfrak{k}(V)$, we could drop the assumption on the rank of the curvature in Corollary 2. Indeed, if (\mathcal{M}, g) is homogeneous, at each point p we have, in addition to the Killing vector fields V, K_1, \dots, K_n spanning V_p^\perp , a Killing vector field \hat{K} transversal to V_p^\perp . In normal Brinkmann coordinates this vector field would have $b = 1$ and hence, by the assumption $\dim(\mathfrak{k}/\mathfrak{k}(V)) = 1$, all the K_i 's would have $a_i = 0$. The proof of Theorem 1 then shows that (\mathcal{M}, g) is a plane wave.

APPENDIX A. NORMAL BRINKMANN COORDINATES FOR PP-WAVES

Here we prove Lemma 3.1. It is well known that, since a pp-wave has a parallel null vector field, it admits local Walker coordinates [25]. Evaluating the curvature condition (3.1) in these coordinates yields the desired form (3.3). We will give some more detail on this, as it gives us the opportunity to describe the coordinate freedom: By the existence of Walker coordinates, there is a x^+ -dependent family of one-forms $\mu = \mu_i(x^+)dx^i$ and a x^+ -dependent family of Riemannian metrics $h = h_{ij}(x^+)dx^i dx^j$ and a smooth function $H = H(x^+, \mathbf{x})$ such that

$$(A.1) \quad g = 2dx^+(dx^- + Hdx^+ + \mu) + h_{ij}dx^i dx^j,$$

or, more conveniently

$$(A.2) \quad g = 2dx^+(dx^- + Hdx^+ + \mu^\top d\mathbf{x}) + d\mathbf{x}^\top h d\mathbf{x},$$

where we set $\mathbf{x} := (x^1, \dots, x^n)$ and slightly abuse the notation when denoting the vector $\mu = (\mu_1, \dots, \mu_n)$ and the matrix $h = (h_{ij})$ by the same symbols as the one form and the metric. Note that the most general coordinate transformation preserving this form is given by

$$(A.3) \quad \tilde{x}^- = \frac{1}{a}x^- + F(x^+, \mathbf{x}), \quad \tilde{\mathbf{x}} = \tilde{\mathbf{x}}(x^+, \mathbf{x}), \quad \tilde{x}^+ = ax^+ + b$$

for constants $a \neq 0$ and b , and a function F of x^+ and the x^i 's. Then, for the new ingredients \tilde{H} , $\tilde{\mu}$ and \tilde{h} of the metric in form (A.2)

$$g = 2d\tilde{x}^+(d\tilde{x}^- + \tilde{H}d\tilde{x}^+ + \tilde{\mu}^\top d\tilde{\mathbf{x}}) + d\tilde{\mathbf{x}}^\top \tilde{h} d\tilde{\mathbf{x}}$$

we get the relations

$$(A.4) \quad \begin{aligned} H &= a(\tilde{H} + \dot{F} + \tilde{\mu}^\top \dot{\tilde{\mathbf{x}}}) + \frac{1}{2}\dot{\tilde{\mathbf{x}}}^\top \tilde{h} \dot{\tilde{\mathbf{x}}} \\ \mu &= a \operatorname{grad}^h(F) + (a\tilde{\mu}^\top + \dot{\tilde{\mathbf{x}}}^\top \tilde{h})D(\tilde{\mathbf{x}}) \\ h &= D(\tilde{\mathbf{x}})^\top \tilde{h} D(\tilde{\mathbf{x}}), \end{aligned}$$

where $\text{grad}^h(F)$ denotes the gradient of F with respect to h , and $D(\tilde{x})$ the Jacobian of \tilde{x} in the x^i directions.

Now we turn to pp-waves. For the curvature of a metric in (A.1) we compute

$$(A.5) \quad R(X, Y)Z = R^h(X, Y)Z + \left((d\nabla^h h(X, Y, Z) - \frac{1}{2}(\nabla_Z^h d\mu)(X, Y) - (R_{X,Y}^h \mu)(Z) \right) \partial_-,$$

for X, Y, Z in the span of the ∂_i 's. Pairing this with ∂_i , condition (3.1) shows that h is a family of flat Riemannian metrics, and hence, by applying a transformation as in (A.3) with $F \equiv 0$, $a = 1$ and $b = 0$ preserving the form of (A.1) but such that $h_{ij} \equiv \delta_{ij}$. In these coordinates, pairing (A.5) with ∂_+ , condition (3.1) becomes

$$0 = \nabla_{\partial_i}^h \mu(\partial_j, \partial_k) = \partial_i(m_{jk}),$$

where $d\mu = M_{ij}dx^i \wedge dx^j$, where the d denotes the differential only in the x^i -directions. Hence $M(x^+) := (M_{ij}(x^+))_{i,j=1}^n \in \mathfrak{so}(n)$ is an x^+ -dependent family of skew-symmetric matrices. For this M , we consider the linear ODE

$$(A.6) \quad \dot{A} = -AM, \quad A(0) = A_- \in O(n).$$

This has a unique solution $A(x^+)$ which satisfies

$$\frac{d}{dx^+}(AA^\top) = -AMA^\top - AM^\top A^\top = 0,$$

since M skew. Hence, $A(0) \in O(n)$ implies that $A(x^+) \in O(n)$ for all x^+ . For such a solution A , we define the x^+ -dependent one-form

$$\alpha = x^\top \dot{A}^\top A d\mathbf{x} = x^l \dot{A}_l^i \delta_{ij} A_k^j dx^k.$$

This form $\alpha + \mu$ is closed,

$$\begin{aligned} d(\mu - \alpha) &= M_{lk} dx^l \wedge dx^k - \dot{A}_l^i \delta_{ij} A_k^j x^l \wedge dx^k \\ &= (M_{lk} - (\dot{A}^\top A)_{lk}) dx^l \wedge dx^k \\ &= (M_{lk} + (M^\top A^\top A)_{lk}) dx^l \wedge dx^k \\ &= 0. \end{aligned}$$

Now, for given μ in (A.1) with $h_{ij} \equiv \delta_{ij}$, let $F = F(x^+, \mathbf{x})$ be a solution to $dF = \mu - \alpha$ and A a solution to (A.6) and consider the coordinate transformation

$$(A.7) \quad \tilde{x}^- = x^- + F(x^+, \mathbf{x}), \quad \tilde{\mathbf{x}} = A\mathbf{x}, \text{ i.e., } \tilde{x}^i = A_k^i x^k, \quad \tilde{x}^+ = x^+.$$

Then, according to (A.4), we have $\tilde{h} = \delta_{ij}$ and moreover,

$$\mu = dF + (\tilde{\mu}^\top + x^\top \dot{A}^\top) A d\mathbf{x} = dF + \tilde{\mu}^\top A d\mathbf{x} + \alpha.$$

Since $dF = \mu - \alpha$, this implies $\tilde{\mu} = 0$, as required. Note that the general transformation preserving the form (3.3) of Brinkmann coordinates are of the form

$$\tilde{x}^- = \frac{1}{a}x^- + F(x^+, \mathbf{x}), \quad \tilde{\mathbf{x}} = A\mathbf{x} + \mathbf{c}(x^+), \quad \tilde{x}^+ = ax^+ + b,$$

where $a \neq 0$ and b are constants, $\mathbf{c}(x^+) \in \mathbb{R}^n$, $A = A(x^+) \in O(n)$ satisfying the PDE

$$(A.8) \quad 0 = a dF + (\dot{A}\mathbf{x} + \dot{\mathbf{c}})^\top A d\mathbf{x}.$$

The integrability condition for this is

$$0 = d\mathbf{x}^\top \dot{A}^\top A d\mathbf{x},$$

which implies that $\dot{A} = 0$ (note that $d\mathbf{x}^\top \dot{A}^\top A d\mathbf{x}$ is indeed a two-form, as $\dot{A}^\top A$ is skew-symmetric). This implies that F is linear in the x^i 's, i.e.,

$$F(x^+, \mathbf{x}) = -\frac{1}{a} \dot{\mathbf{c}}^\top(x^+) A \mathbf{x} + \beta(x^+)$$

for a function $\beta = \beta(x^+)$. Hence, the general transformation preserving the form (3.3) of a Brinkmann coordinates are given by a constant matrix $A \in \text{O}(n)$, a vectorial function \mathbf{c} of x^+ and a real function β of x^+ , and two real numbers $a \neq 0$ and b , and the transformation is

$$(A.9) \quad \tilde{x}^- = \frac{1}{a}(x^- - \dot{\mathbf{c}}(x^+)^\top A \mathbf{x}) + \beta(x^+), \quad \tilde{\mathbf{x}} = A \mathbf{x} + \mathbf{c}(x^+), \quad \tilde{x}^+ = ax^+ + b,$$

The function \tilde{H} is then given as

$$(A.10) \quad \tilde{H} = \frac{1}{a}(H + \ddot{\mathbf{c}}(x^+)^\top A \mathbf{x}) + \dot{\beta} - \frac{1}{2a} \dot{\mathbf{c}}(x^+)^\top \dot{\mathbf{c}}(x^+)$$

Clearly, by applying a translation we can choose these coordinates in a way that p goes to the origin.

It remains to show that for a given Brinkmann coordinates $\varphi = (x^+, x^-, \mathbf{x})$ mapping p to the origin, there is a coordinate transformation of the form (A.9) that fixes the origin and provides us with normal Brinkmann coordinates, i.e., for which the new function \tilde{H} satisfies

$$(A.11) \quad \begin{aligned} \tilde{H}|_{\tilde{\varphi}^{-1}(x^+, \mathbf{0})} &= 0 \\ \frac{\partial \tilde{H}}{\partial \tilde{x}^i}|_{\tilde{\varphi}^{-1}(x^+, \mathbf{0})} &= 0 \end{aligned}$$

for all x^+ . To this end we consider a transformation (A.9) with $A = \delta_{ij}$, $b = 0$ and $a = 1$. Let $\mathbf{c} = (c_1, \dots, c_n)$ the solution to the ODE system

$$\ddot{c}_i(t) = -\frac{\partial}{\partial x^i} H(\varphi^{-1}(t, -\mathbf{c}(t))),$$

for $i = 1, \dots, n$ with one initial condition $c_i(0) = 0$. Given such a solution $\mathbf{c} = (c_1, \dots, c_n)$, let β be the solution to the ODE

$$\dot{\beta} = \frac{1}{2} \dot{\mathbf{c}}^\top \dot{\mathbf{c}} - H(\varphi^{-1}(t, -\mathbf{c}(t))),$$

with the initial condition $\beta(0) = 0$. Using these solutions \mathbf{c} and β in the coordinate transformation (A.9), the formula (A.10) shows that in the new coordinates we have equations (A.11) for all x^+ .

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SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF ADELAIDE, SA 5005, AUSTRALIA
 E-mail address: wolfgang.globke@adelaide.edu.au, thomas.leistner@adelaide.edu.au